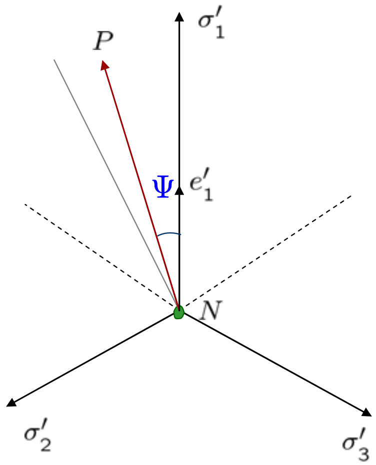
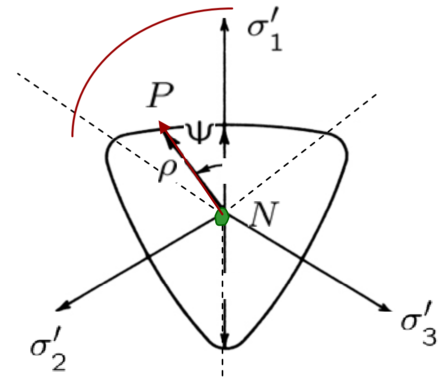
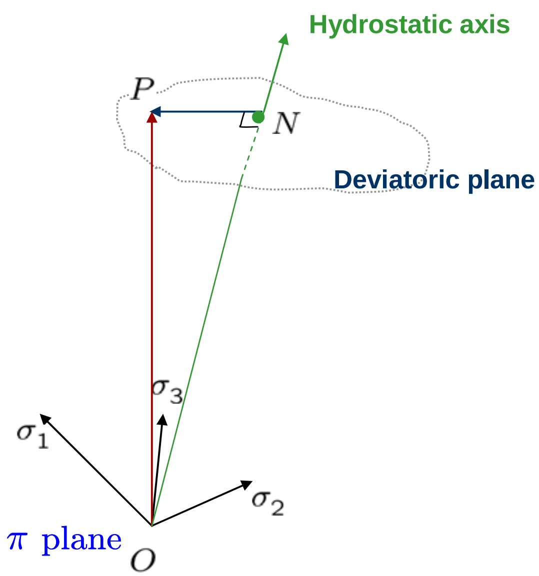
Constitutive models: Failure Surface



$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right)\ ,\ \ \rho = \sqrt{2J_2}$$

$$p\ =\ const$$

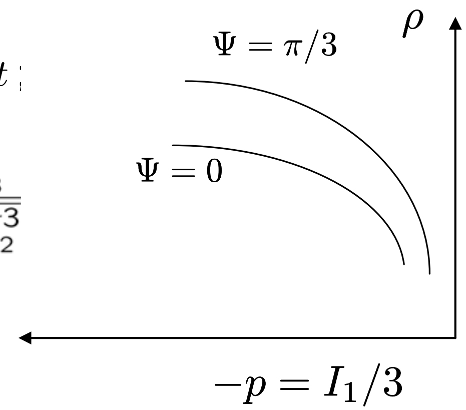




$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right)\ ,\ \ \rho = \sqrt{2J_2}$$

$$\Psi\ =\ const$$

$$\cos 3\Psi = \frac{3\sqrt{3}}{2}\frac{J_3}{\sqrt{J_2^3}}$$



$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2}$$

$$\Psi = const\,;\ \Psi = 0\,, \Psi = \pi/3$$ meridian plane

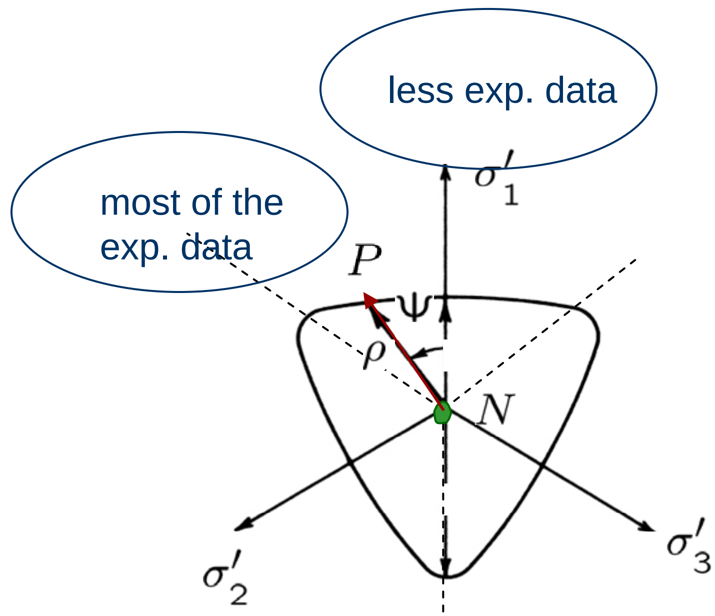
Copmressive meridian:

$$\sigma_1 = \sigma_2 > \sigma_3$$

$$\Downarrow$$

triaxial compression:
hydrostatic pressure
+ axial compression

$$\Psi = \pi/3$$



Tensile meridian:

triaxial tension:
hydrostatic pressure
+ axial tension

$$\sigma_3 = \sigma_2 < \sigma_1$$

$$\Downarrow$$

$$\Psi = 0$$

Shear meridian:

$$\sigma_1\,,\ \frac{\sigma_1+\sigma_3}{2}\,,\ \sigma_3 \qquad \Psi = \pi/6$$

$$f\left(\sigma_{ij}\right) = 0$$

Historical overview

Hypothesis 1: the theory of maximum normal stress states: the material strength is completely determined by the stress and the two strength limits in tension, σ_t and compression, σ_c .

Galilei, Leibniz (17c)

$$\sigma_3 = -\sigma_c \qquad \sigma_1 \leq \sigma_t$$

$$\sigma_3 \geq -\sigma_c \qquad \sigma_1 = \sigma_t$$

Bauschinger experiments 1874 on steel with diffrent carbon content -> the limit of elastic response in torsion is appr. the half of the elastic limit in tension

Hypothesis 2: theory of maximum linear relative deformation (assess material strength through max tensile elongation (*te*))

$$\sigma_1 - k(\sigma_2 + \sigma_3) \leq \sigma_{te}$$

$$\sigma_3 - k(\sigma_1 + \sigma_2) \geq -\sigma_c$$

coefficient of lateral compression deformation:

$$k = 1/4$$

$\Rightarrow$ **strength in pure compression exceeds 4 times the strength in tension**

$f\left(\sigma_{ij}\right) = 0$

Historical overview

Hypothesis 3: the theory of maximum difference in normal stress in which the material strength is defined by the maximum shear stress;
- only maximum and minimum principal stresses play a role in this theory.

Coulomb 1773

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) \leq \frac{\sigma_{te}}{2}$$

Pure shear:

$$\tau_{max} = \sigma_1 = -\sigma_3 \leq \frac{\sigma_{te}}{2}$$

(confirmed by the Bauschinger experiment)

Duguet 1882

assumed like Coulomb that the failure is due to shear;
- resistance to shear failure depends on cohesion and internal friction. The later value changes with change of the normal stress, σ_ν , acting on the shear plane.

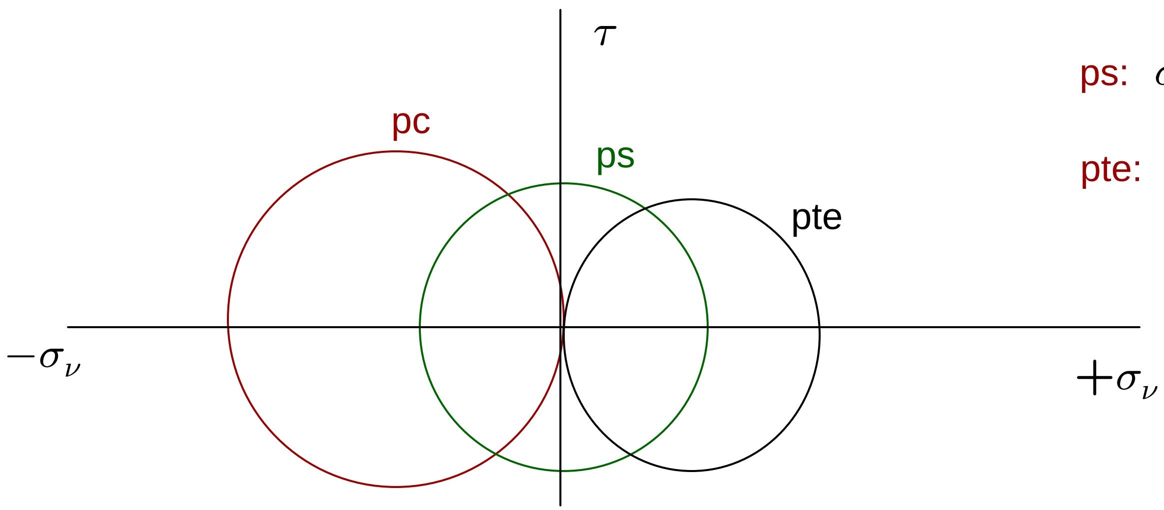
$$\tau + f\,\sigma_\nu = const \qquad f = 0.176$$

coefficient of internal friction

$$f\left(\sigma_{ij}\right) = 0$$

Historical overview

Mohr 1882; 1900; Mohr‘s principal circles -> Mohr‘s parabola as a hull for failure states



pc: $\sigma_2 = \sigma_1 = 0 \quad \sigma_3 = -\sigma_c$

ps: $\sigma_2 = 0 \quad \sigma_1 = -\sigma_3 = \sigma_s$

pte: $\sigma_3 = \sigma_2 = 0 \quad \sigma_1 = \sigma_{te}$

$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2}$$

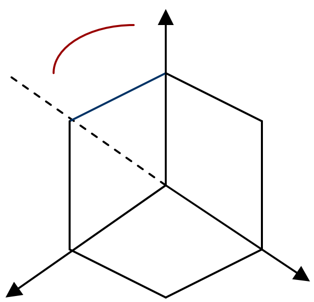
One parameter failure models

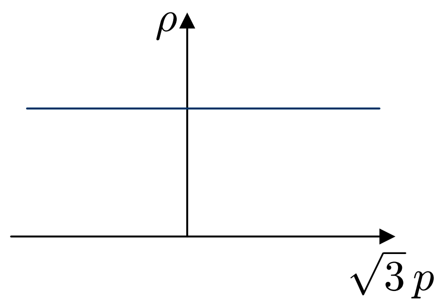
Tresca (1868) shear stress criterion:

$$f(\rho,\ \Psi) = \rho \sin\left(\Psi + \frac{\pi}{3}\right) - \sqrt{2}\,k = 0 \qquad \Psi \in [0, \pi/3]$$

for metals the hydrostatic pressure influence can be neglected

The material is at yield/failure when the maximum shear reaches a certain limit value k :





$$\sigma_1 - \sigma_3 = 2\,k$$

$$\sigma_1 = \max\{\sigma_i,\ i = 1,2,3\}$$

$$\sigma_3 = \min\{\sigma_i,\ i = 1,2,3\}$$

Reuß (1933):

$$f(I_1,\ J_2,\ J_3) = 4\,J_2^3 - 27\,J_3^2 - 36\,k^2 J_2^2 + 96\,k^4 J_2 - 64 k^6 = 0$$

Tresca (1868) shear stress criterion:

$$f(\rho, \Psi) = \rho \sin\left(\Psi + \frac{\pi}{3}\right) - \sqrt{2}\,k = 0$$

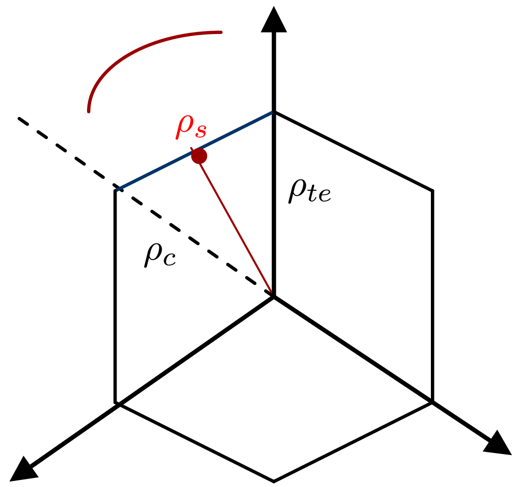
Pure tension (te)

$$\Psi = 0 \longrightarrow \rho_{te} = \frac{\sqrt{2}\,k}{\sin\frac{\pi}{3}} \longrightarrow \rho_{te} = \frac{2\sqrt{2}\,k}{\sqrt{3}}$$

$$\rho_{te} = \sqrt{2\,J_2} = \sqrt{\left(\frac{4}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2\right)} = \sqrt{\frac{2}{3}}\,\sigma$$

$$\Downarrow$$

at failure $\sigma = 2\,k$



Pure compression (c)

$$\Psi = \frac{\pi}{3} \longrightarrow \rho_c = \frac{\sqrt{2}\,k}{\sin\frac{2\pi}{3}} \longrightarrow \rho_c = \frac{2\sqrt{2}\,k}{\sqrt{3}}$$

Pure shear (s)

$$\Psi = \frac{\pi}{6} \longrightarrow \rho_s = \frac{\sqrt{2}\,k}{\sin\frac{\pi}{2}} \longrightarrow \rho_s = \sqrt{2}\,k$$

$$\rho_s = \sqrt{2\,J_2} = \sqrt{(\tau^2 + 0 + \tau^2)} = \sqrt{2}\,\tau$$

$$\sigma_1 - \sigma_3 = 2\tau$$

at pure shear failure

$$\tau = k$$

Constitutive models: Failure Models

$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2}$$

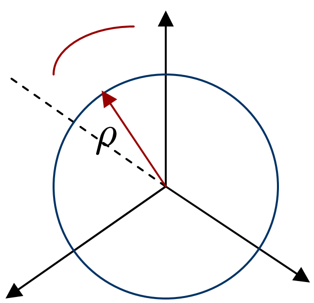
One parameter failure models

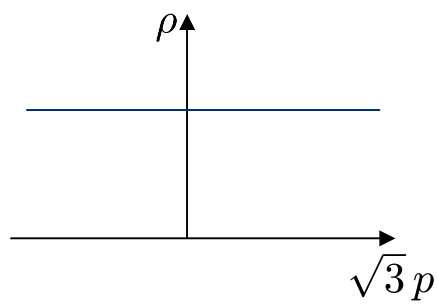
von Mises (1913) shear stress criterion: the onset of yielding/failure does not depend on the first stress invariant (volumetric stress) - J_2 plasticity

$$f\left(\rho\right) = \rho - \sqrt{2}\,k = 0$$

↓

magnitude of the shear stress at yielding in pure shear





The material yields when the distance of the corresponding stress point from the hydrostatic axis in the principal stress space reaches certain limit -> failure curve is a cyrcle

since p has no influence -> meridians II to the hydrostatic axis

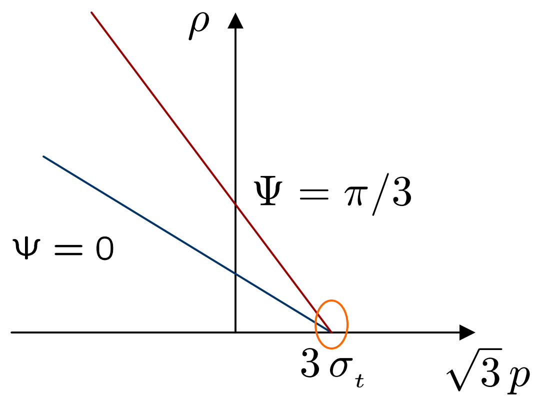
cylinder in principal stress space

$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right)=0\,,\quad \rho=\sqrt{2J_2}\qquad\qquad \Psi\in[0,\pi/3]$$

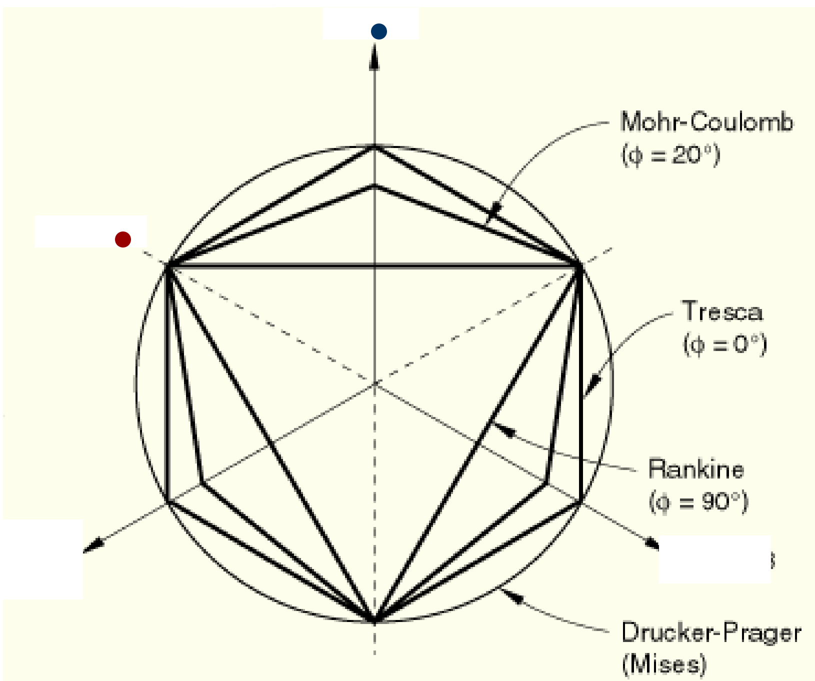
Rankine Criterion: for modeling cracking of concrete (limited tensile capacity)

Yielding/failure starts when maximum principal stress reaches the tensile yield/failure stress -> it gives the triangular Rankine pyramid in the principal stress space

$$\sqrt{2}\,\rho\cos\Psi+\sqrt{3}\,p-\sqrt{3}\sigma_t=0$$



tension cutoff



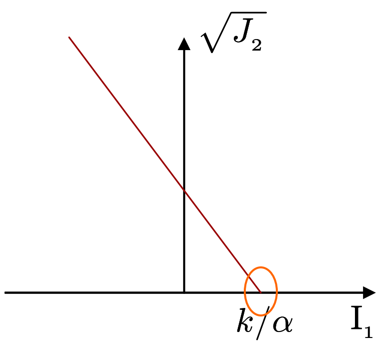
$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right) = 0\,,\;\; \rho = \sqrt{2J_2}$$

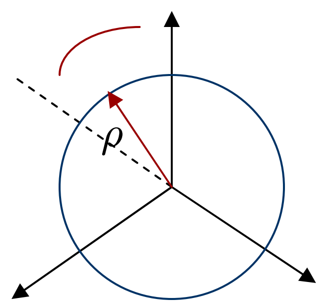
One parameter failure models

Burzinski (1929) (B) – Drucker-Prager (1952) (D) model: modelling failure for materials with internal friction: sand, soil, rock, concrete. Pressure sensitive failure (first invariant included in the model). Analogue to von Mises criterion but the value of the shear yield is adjusted according to the hydrostatic stress failure feature -> conical failure surface

$$f\left(I_1,\, J_2\right) = c_1\, I_1 + c_2\, I_1^2 + c_3\, J_2 - 1 = 0 \qquad \text{(B)}$$

$$f\left(I_1,\, J_2\right) = \alpha\, I_1 + \sqrt{J_2} - k = 0 \qquad \text{(D-P)}$$





$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\ \ \rho = \sqrt{2J_2}$$

One parameter failure models

Gurson (1977) model: includes porosity (modeling powder compacted metals)

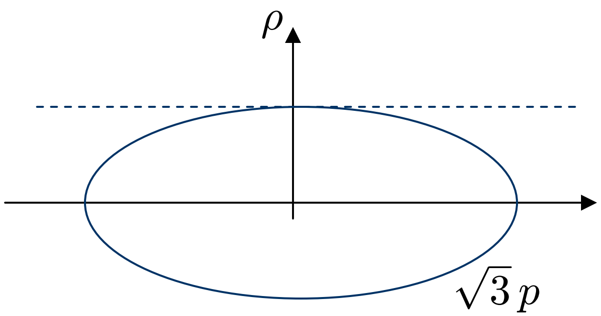
$$f\left(I_1,\ J_2,\ n\right) = \frac{3\,J_2}{k^2} + 2\,n\,\cosh\frac{I_1}{2k} - 1 - n^2 = 0$$

In the meridian plane is ellipsoid

k - uniaxial yield/failure stress of the basic material (without pores)

n - volume fraction of the voids (porosity)

Gurson model converges to von Mises model for $n \rightarrow 0$



$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right)=0\,,\quad \rho=\sqrt{2J_2}$$

Two parameters failure models

sensitive to hydrostatic states of stress + assumption of geometrical similarity of the deviatoric cross sections of the failure surface

Mohr- Coulomb failure model:

$$|\tau| = c - \sigma_\nu \tan\phi$$

Failure is assumed to occure when the shear stress τ on any plane at a point in the material reaches a value that depends linearly upon the normal stress (σ_ν) in the same plane

$$f\left(p,\ J_2,\ \Psi\right)=p\sin\phi+\sqrt{J_2}\sin\left(\Psi+\frac{\pi}{3}\right)+\frac{\sqrt{J_2}}{\sqrt{3}}\cos\left(\Psi+\frac{\pi}{3}\right)\sin\phi-c\cos\phi=0$$

$$\Psi\in[0,\pi/3]$$

Constitutive models: Failure Models

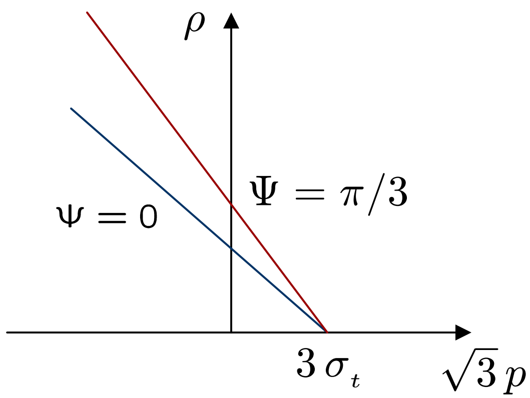
$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2} \qquad\qquad \Psi \in [0, \pi/3]$$

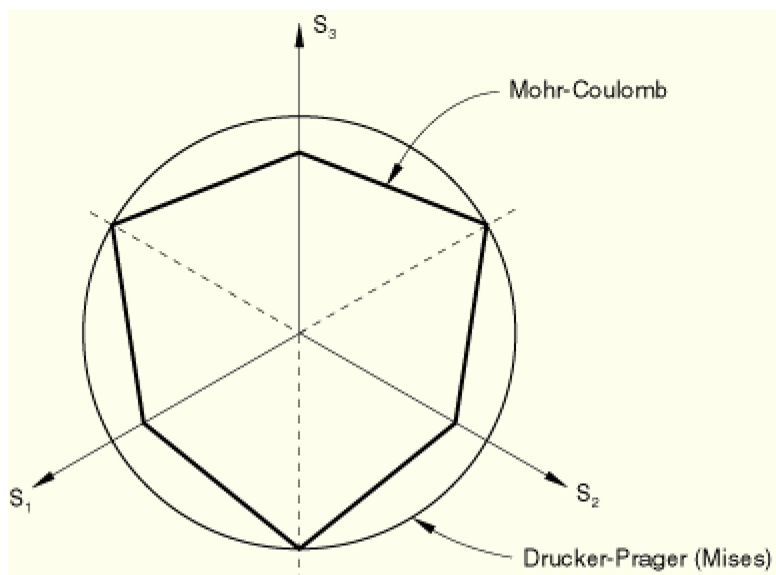
$$f\,(\rho) = \rho - \sqrt{2}\,k = 0$$

von Mises

$$f\,(p,\ J_2,\ \Psi) = p\,\sin\phi + \sqrt{J_2}\,\sin\left(\Psi + \frac{\pi}{3}\right) + \frac{\sqrt{J_2}}{\sqrt{3}}\,\cos\left(\Psi + \frac{\pi}{3}\right)\sin\phi - c\,\cos\phi = 0$$

Mohr- Coulomb





$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right) = 0\,,\;\; \rho = \sqrt{2J_2} \qquad\qquad \Psi \in [0, \pi/3]$$

$$f\,(\rho,\, \Psi) \;=\; \rho \sin\left(\Psi + \frac{\pi}{3}\right) - \sqrt{2}\,k = 0$$ Tresca

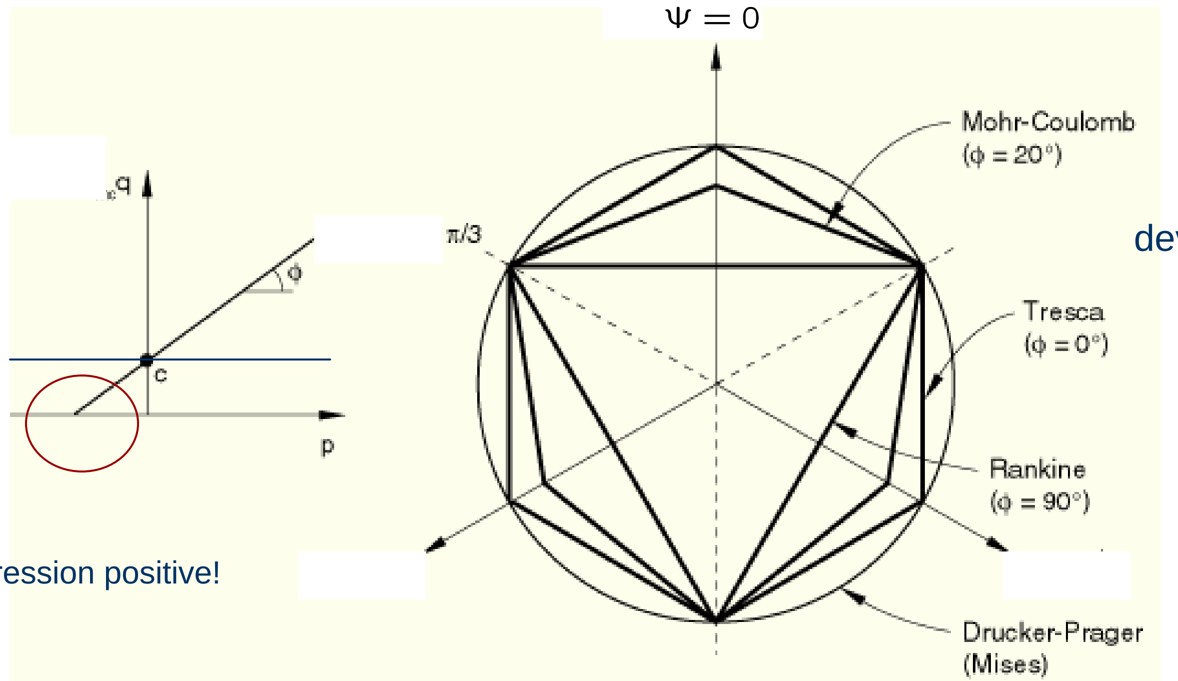
von Mises

$$f\,(\rho) \;=\; \rho - \sqrt{2}\,k = 0$$

$$f\,(p,\, J_2,\, \Psi) \;= p \sin\phi + \sqrt{J_2}\,\sin\,(\Psi + \frac{\pi}{3}) + \frac{\sqrt{J_2}}{\sqrt{3}}\cos\,(\Psi + \frac{\pi}{3})\,\sin\phi - c\,\cos\phi = 0$$

Mohr- Coulomb

meridian plane



deviatoric plane

compression positive!

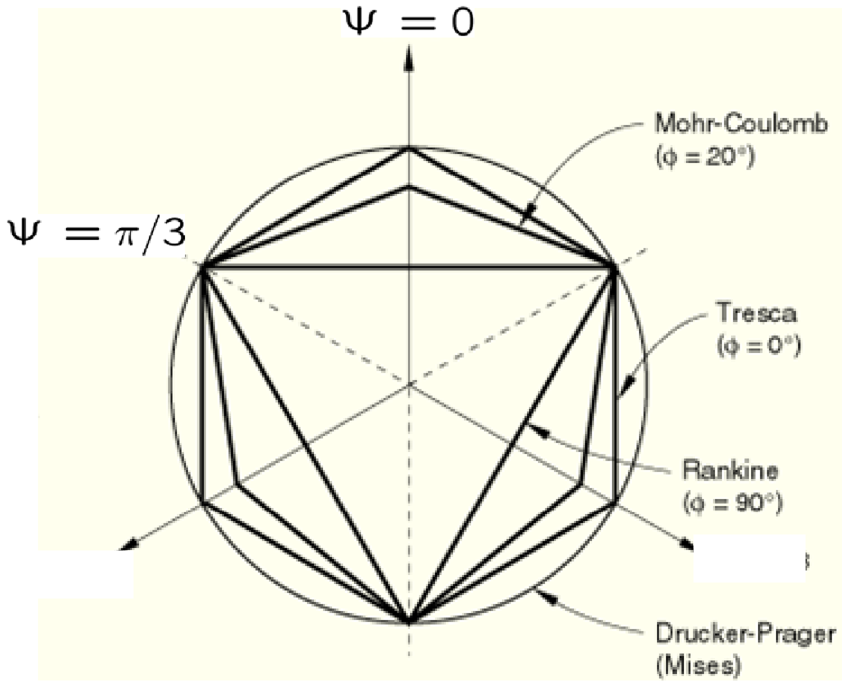
$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2} \qquad\qquad \Psi \in [0, \pi/3]$$

$$f\,(\rho,\ \Psi) = \rho \sin\left(\Psi + \frac{\pi}{3}\right) - \sqrt{2}\,k = 0$$ Tresca

$$f\,(\rho) = \rho - \sqrt{2}\,k = 0$$ von Mises

$$f\,(p,\ J_2,\ \Psi) = p \sin\phi + \sqrt{J_2}\,\sin(\Psi + \frac{\pi}{3}) + \frac{\sqrt{J_2}}{\sqrt{3}}\cos(\Psi + \frac{\pi}{3})\,\sin\phi - c\cos\phi = 0$$

Mohr- Coulomb



$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right) = 0\,,\;\; \rho = \sqrt{2J_2} \qquad \Psi \in [0, \pi/3]$$

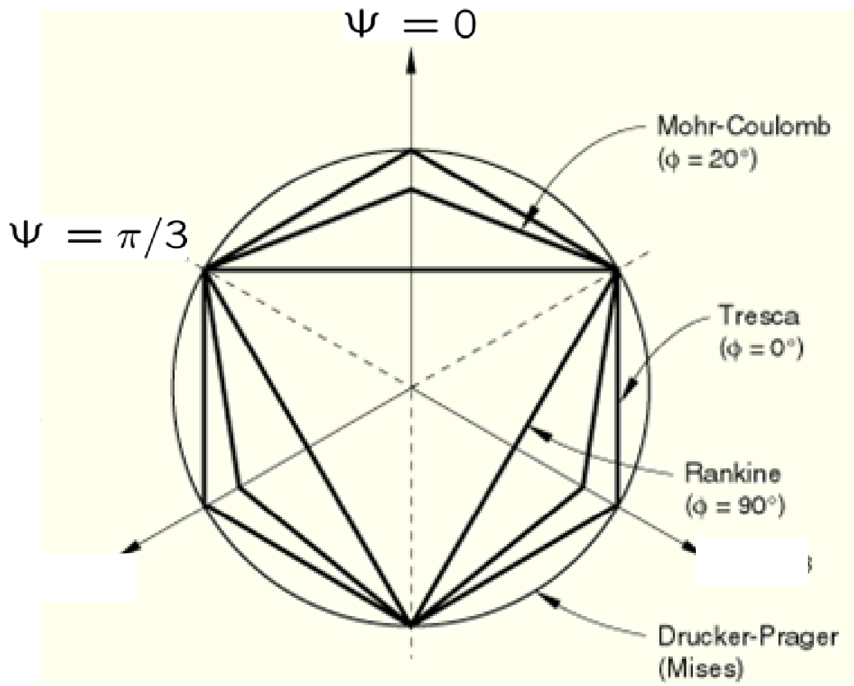
$$f\,(\rho,\, \Psi) \;=\; \rho \sin\left(\Psi + \frac{\pi}{3}\right) - \sqrt{2}\,k = 0$$ Tresca

von Mises

$$f\,(\rho) \;=\; \rho - \sqrt{2}\,k = 0$$

$$f\,(p,\, J_2,\, \Psi) \;= p \sin\phi + \sqrt{J_2}\,\sin\,(\Psi + \frac{\pi}{3}) + \frac{\sqrt{J_2}}{\sqrt{3}}\,\cos\,(\Psi + \frac{\pi}{3})\,\sin\phi - c\,\cos\phi = 0$$

Mohr- Coulomb



Plasticity is the property of the solid body to deform under applied external force and to possess permanent or temporal residual deformation after the applied load is removed.

Main feature of plasticity: $\sigma \sim \varepsilon$

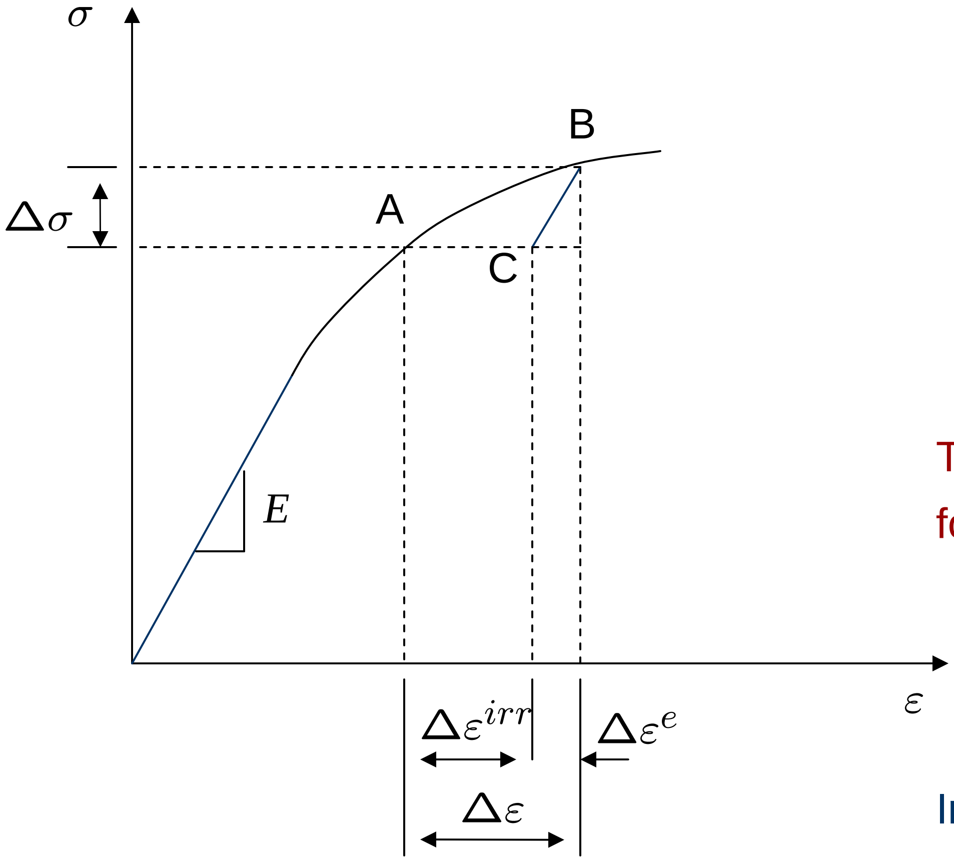
is not uniquely determined by the current state.

Elastic state of a solid body is a state at which an independent of time uniquely determined relationship between stresses and strains exists for any given temperature.

Plastic state of a solid body is a state at which for a given temperature the relationship between stresses and strains at each moment of time becomes uniquely determined if at least one (or all) preceding stress-strain state and the corresponding to it temperature are known.

Otherwise $\sigma \sim \varepsilon$ is not determined.

1D ⟷ 3D



uniaxial test

- Reversible elastic deformation
- Irreversible, inelastic deformation

Term plastic deformation will be used for inelastic time – independent strain

Inelastic deformation may be:

– creep (time dependent);

– viscoplastic (rate dependent)