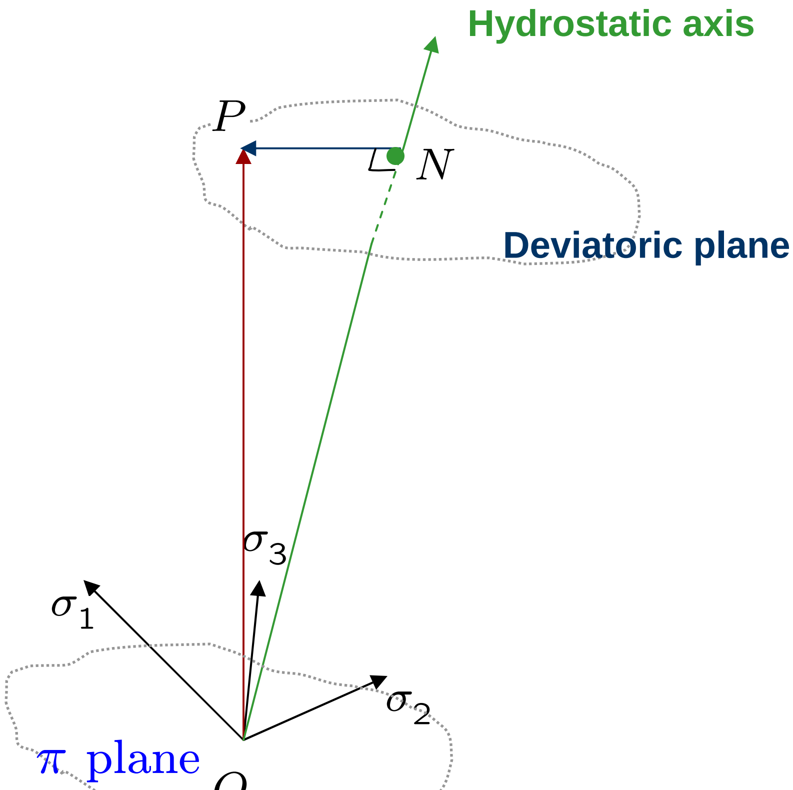
Constitutive models: Geometrical interpretation of stress invariants



$$P(\sigma_1, \sigma_2, \sigma_3) \qquad N(p, p, p)$$

$$N\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}, \frac{\sigma_1+\sigma_2+\sigma_3}{3}, \frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)$$

$$\overrightarrow{ON} = (p, p, p)$$

$$\|\overrightarrow{ON}\| = \sqrt{\overrightarrow{ON} \cdot \overrightarrow{ON}} = \sqrt{3}\,p$$

$$\|\overrightarrow{NP}\| = \sqrt{\overrightarrow{PN} \cdot \overrightarrow{PN}} = \sqrt{2\,J_2}$$

$$\overrightarrow{NP} = (s_1, s_2, s_3)$$

$$\Downarrow$$

$$\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$$

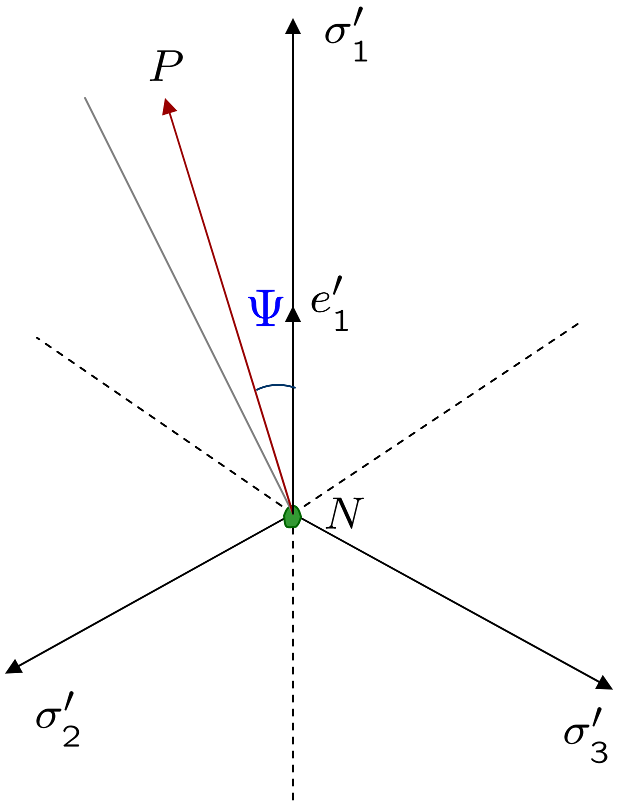
$$\sqrt{2\,J_2}$$

$$p$$

Deviatoric plane equation:

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\,p \quad \Rightarrow \quad N \in \Pi, \quad P \in \Pi$$

+ the deviatoric plane is perpendicular to the hydrostatic axis



Projection onto the deviatoric plane

$$P(\sigma_1, \sigma_2, \sigma_3) \qquad N(p, p, p)$$

$$e'_1 = \frac{1}{\sqrt{6}}(2, -1, -1)$$

$$\overrightarrow{NP} \cdot \overrightarrow{e}'_1 = \|\overrightarrow{NP}\| \cos\Psi = \sqrt{2J_2} \cos\Psi$$

$$\Rightarrow \quad \cos\Psi = \frac{\overrightarrow{NP} \cdot \overrightarrow{e}'_1}{\sqrt{2J_2}} = \frac{1}{\sqrt{2J_2}} \frac{1}{\sqrt{6}} (2s_1 - s_2 - s_3)$$

with $-s_2 - s_3 = s_1$

$$\Downarrow$$

$$\cos\Psi = \frac{\sqrt{3}}{2} \frac{s_1}{\sqrt{2J_2}} \qquad \Psi \in [0, \pi/3]$$

$$\cos 3\Psi = 4\cos^3\Psi - 3\cos\Psi \Rightarrow$$

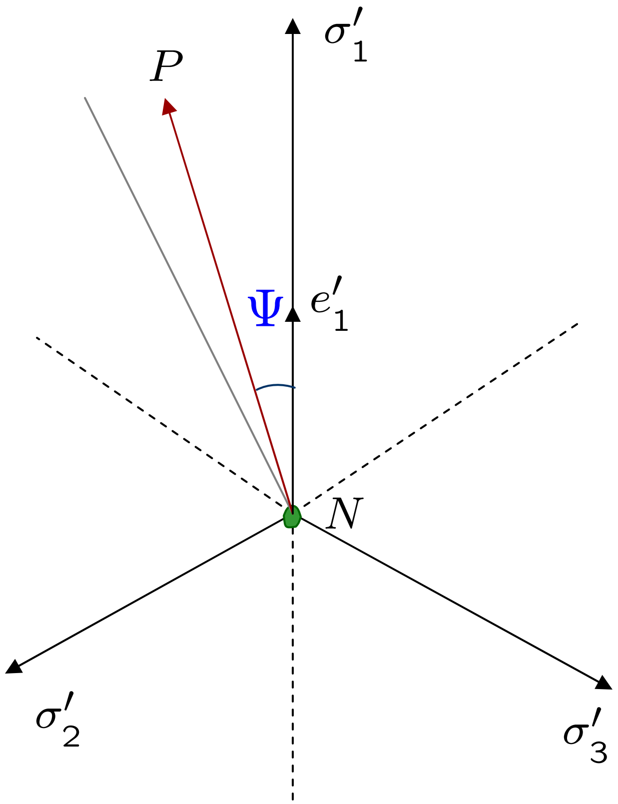
$$\cos 3\Psi = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}}$$

Lode‘s angle

p

$\sqrt{2\,J_2}$

Ψ



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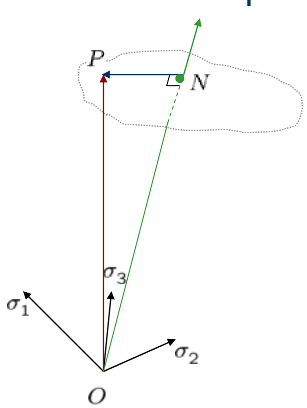

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Lode's angle

$$p$$
$$\sqrt{2J_2}$$
$$\Psi$$

Complete progressive analysis of the structure up to collapse

- concrete reactor vessels;
- nuclear containment strucutures;
- parts of offshore platforms ...

Experimental study is very expensive and therefor modelling is of importance

Most commonly used material laws consist of:

theory of elasticity + criteria defining failure

↓

Failure surface (constitutive function)

$$f\left(\sigma_{ij}\,,\,T\,,\,t\,,\,\varpi\right)$$

$$f\left(\sigma_{ij}\right)$$

$$f\left(\sigma_1,\,\sigma_2,\,\sigma_3\right)$$

$$f\left(I_1,\,J_2,\,J_3\right)$$

$$f\left(\sqrt{3}p,\,\rho,\,\Psi\right)\ ,\ \ \rho=\sqrt{2J_2}$$

p

ρ

Ψ

$f\left(\sigma_{ij}\right)$ $\quad$ $f\left(\sigma_1,\, \sigma_2,\, \sigma_3\right)$

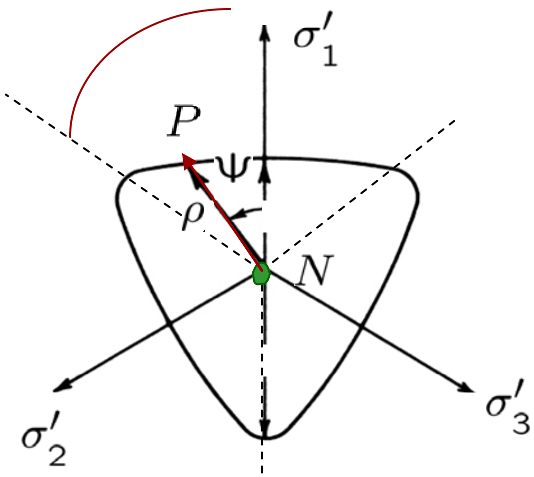
$f\left(I_1,\, J_2,\, J_3\right)$

$f\left(\sqrt{3}p,\, \rho,\, \Psi\right)\ ,\quad \rho = \sqrt{2J_2}$

Failure surface (constitutive function)

Geometrical interpretation:

- Failure curve – the cross section of the failure surface and a deviatoric plane



$p = const$

- Meridians of the failure surface – the intersection curves between the failure surface and a plane (meridian plane) containing the hydrostatic axis

$\Psi = const$

Failure curve (FC) characteristics (based on exprerimantal results):

$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right) \;,\; \rho = \sqrt{2J_2}$$

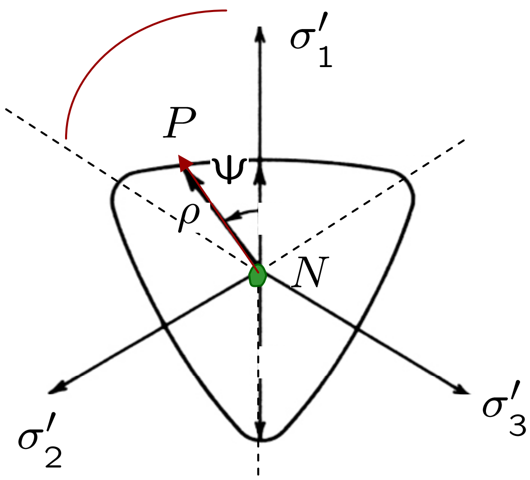
$$p = const$$

- The FC is smooth.
- The FC is convex:

$$\frac{\partial^2 \rho}{\partial \Psi^2} < \rho + \frac{2}{\rho}\left(\frac{\partial \rho}{\partial \Psi}\right)$$

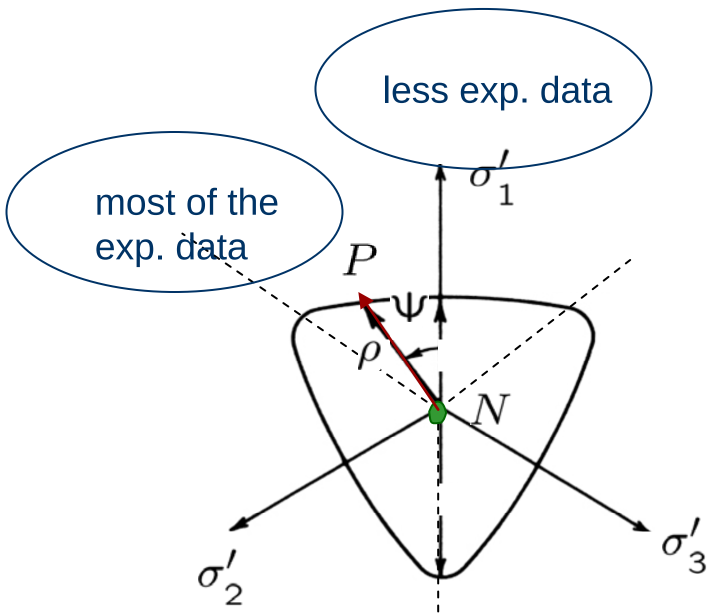
- The FC is nearly triangular for small hydrostatic pressures and becomes more circular with increasing p .

(FC is nonaffine)



$$f\left(\sqrt{3}p,\ \rho,\ \Psi\right) = 0\,,\quad \rho = \sqrt{2J_2}$$

$$\Psi = const\,;\ \Psi = 0\,,\Psi = \pi/3$$ meridian plane



Copmressive meridian:

$$\sigma_1 = \sigma_2 > \sigma_3$$

$$\Downarrow$$

triaxial compression:
hydrostatic pressure
+ axial compression

$$\Psi = \pi/3$$

Tensile meridian:

triaxial tension:
hydrostatic pressure
+ axial tension

$$\sigma_3 = \sigma_2 < \sigma_1$$

$$\Downarrow$$

$$\Psi = 0$$

Shear meridian:

$$\sigma_1\,,\ \frac{\sigma_1+\sigma_3}{2}\,,\ \sigma_3$$

$$\Psi = \pi/6$$

Copmressive meridian: one dimentional compression

$$\sigma_3 = -\sigma \qquad \sigma_1 = \sigma_2 = 0$$

$$3\,p = \sigma_{ij}\,\delta_{ij} = -\sigma$$

$$s_1 = \sigma_1 - p = \frac{\sigma}{3} \qquad s_2 = s_1 \qquad s_3 = \sigma_3 - p = -\frac{2}{3}\sigma$$

$$J_2 = \frac{1}{2}\,s_{ij}\,s_{ij} \longrightarrow J_2 = \frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right) = \frac{1}{3}\,\sigma^2$$

$$J_3 = \frac{1}{3}\,s_{ij}\,s_{jk}s_{ki} \rightarrow J_3 = \frac{1}{3}\left(s_1^3 + s_2^3 + s_3^3\right) = -\frac{2}{27}\,\sigma^3$$

$$\cos 3\Psi = \frac{3\sqrt{3}}{2}\,\frac{J_3}{\sqrt{J_2^3}} \rightarrow \cos 3\Psi = -\frac{3\sqrt{3}}{2}\,\frac{2}{27}\sigma^3\,\frac{\sqrt{3^3}}{\sigma^3} = -1$$

$$3\Psi = \pi$$

$$P(\sqrt{3}p, \rho, \Psi) \rightarrow P(-\frac{\sqrt{3}}{3}\sigma, \sqrt{\frac{2}{3}}\,\sigma, \frac{\pi}{3})$$

1D compression

Failure surfece characteristics in the meridian planes

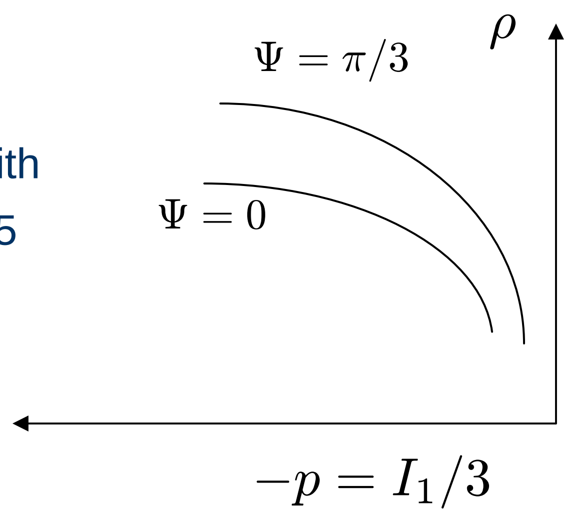
$$f\left(\sqrt{3}p,\, \rho,\, \Psi\right) \;,\;\; \rho = \sqrt{2J_2}$$

$$\Psi \;=\; const\,;$$

- The FC depends on the hydrostatic component of stress, I_1 .
- The meridians are curved, smooth and convex
- Tensile meridian is posed below the compressive one

•The tensile and compressive meriadians tend to coinside with increasing the hydrostatic pressure (the ratio starting with 0.5 near the π-plane)

- Pure hydrostatic loading can not cause failure



$$f\left(\sigma_{ij}\right) = 0$$

Historical overview

Hypothesis 1: the theory of <u>maximum normal stress</u> states: the material strength is completely determined by the stress and the two strength limits in tension, σ_t and compression, σ_c .

Galilei, Leibniz (17c)

$$\sigma_3 = -\sigma_c \qquad \sigma_1 \leq \sigma_t$$

$$\sigma_3 \geq -\sigma_c \qquad \sigma_1 = \sigma_t$$

Bauschinger experiments 1874 on steel with diffrent carbon content -> the limit of elastic response in torsion is appr. the half of the elastic limit in tension

Hypothesis 2: theory of <u>maximum linear relative deformation</u> (assess material strength through max tensile elongation (*te*))

$$\sigma_1 - k(\sigma_2 + \sigma_3) \leq \sigma_{te}$$

$$\sigma_3 - k(\sigma_1 + \sigma_2) \geq -\sigma_c$$

coefficient of lateral compression deformation:

$$k = 1/4$$

$\Rightarrow$ **strength in pure compression exceeds 4 times the strength in tension**

$f\left(\sigma_{ij}\right) = 0$

Historical overview

Hypothesis 3: the theory of maximum difference in normal stress in which the material strength is defined by the maximum shear stress;
- only maximum and minimum principle stresses play a role in this theory.

Coulomb 1773

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) \leq \frac{\sigma_{te}}{2}$$

Pure shear:

$$\tau_{max} = \sigma_1 = -\sigma_3 \leq \frac{\sigma_{te}}{2}$$

(confirmed by the Bauschinger experiment)

Duguet 1882

assumed like Coulomb that the failure is due to shear;
- resistance to shear failure depends on cohesion and internal friction. The later value changes with change of the normal stress, σ_ν , acting on the shear plane.

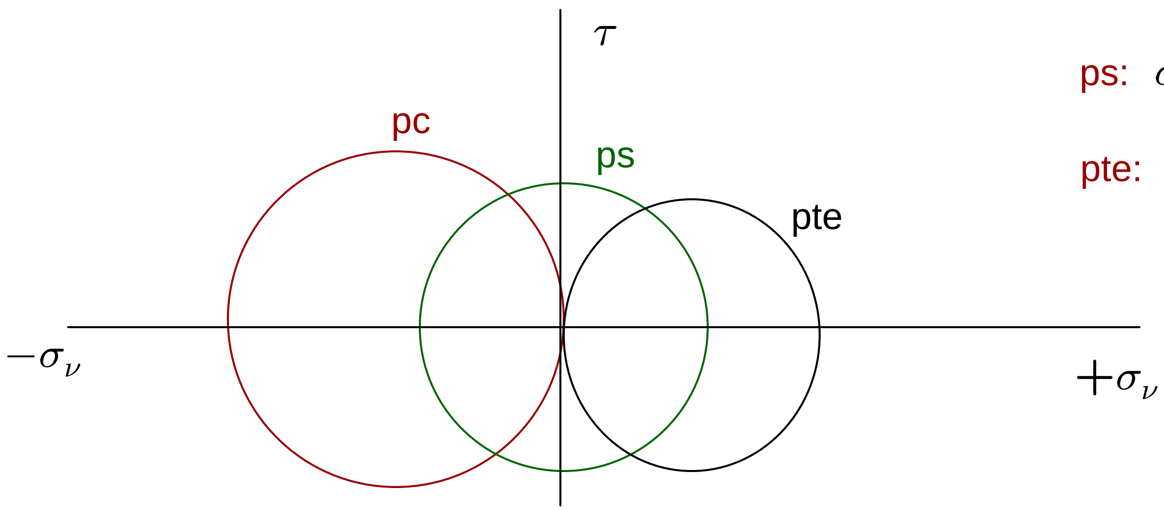
$$\tau + f\,\sigma_\nu = const \qquad f = 0.176$$

coefficient of internal friction

$f\left(\sigma_{ij}\right) = 0$

Historical overview

Mohr 1882; 1900; Mohr's principle circles -> Mohr's parabola as a hull for failure states



pc: $\sigma_2 = \sigma_1 = 0 \quad \sigma_3 = -\sigma_c$

ps: $\sigma_2 = 0 \quad \sigma_1 = -\sigma_3 = \sigma_s$

pte: $\sigma_3 = \sigma_2 = 0 \quad \sigma_1 = \sigma_{te}$