Example 5: an incremental relation based on hyperelasticity

$$W = W(\theta, \epsilon)$$

strain energy density function

$$\theta = \varepsilon_{kk}\,; \qquad \epsilon = \sqrt{\frac{2}{3} e_{ij} e_{ij}}$$

$$\Downarrow$$

$$p = \frac{\partial W}{\partial \theta} \qquad q = \frac{\partial W}{\partial \epsilon}$$

and

$$p = p(\theta, \epsilon) \qquad q = q(\theta, \epsilon)$$

$$\Downarrow$$

$$K_t = \frac{\partial p}{\partial \theta} = \frac{\partial^2 W}{\partial \theta^2} \qquad 3G_t = \frac{\partial q}{\partial \epsilon} = \frac{\partial^2 W}{\partial \epsilon^2} \qquad R = \frac{\partial p}{\partial \epsilon} = \frac{\partial q}{\partial \theta} = \frac{\partial^2 W}{\partial \epsilon \partial \theta}$$

$$\begin{pmatrix} dp \\ dq \end{pmatrix} = \begin{pmatrix} K_t & \boxed{R} \\ \boxed{R} & 3G_t \end{pmatrix} \begin{pmatrix} d\theta \\ d\epsilon \end{pmatrix}$$

Loading and unloading behaviour

Special class of hypoelastic materials with the following features:

- Incrementally isotropic behaviour;
- Different material response in loading and unloading.

$$\dot{p} = K_t\, \dot{\varepsilon}_{kk} \qquad \dot{s}_{ij} = 2G_t \dot{e}_{ij}\,; \qquad 3\,\dot{p} = \dot{\sigma}_{kk}$$

K_t and G_t are functions of stress and differ in primary loading and subsequent unloading and reloading.

This invokes the need of *loading – unloading – reloading* conditions that are **invariant** with respect to coordinate transformations:

Models involving additional functions of stress invariants.

$$\dot{p} = K_t \, \dot{\varepsilon}_{kk} \qquad \dot{s}_{ij} = 2G_t \dot{e}_{ij} \, ; \qquad 3\, \dot{p} = \dot{\sigma}_{kk}$$

loading – unloading – reloading conditions :

$$K_t = K_{tL} \quad \mapsto \quad p = p^f \ \cup \ \dot{p} > 0 \qquad G_t = G_{tL} \quad \mapsto \quad J_2 = J_2^f \ \cup \ \dot{J}_2 > 0$$

$$K_t = K_{tU} \quad \mapsto \quad p \leq p^f \ \cup \ \dot{p} < 0 \qquad G_t = G_{tU} \quad \mapsto \quad J_2 \leq J_2^f \ \cup \ \dot{J}_2 < 0$$

$$K_t = K_{tR} \quad \mapsto \quad p < p^f \ \cup \ \dot{p} > 0 \qquad G_t = G_{tR} \quad \mapsto \quad J_2 < J_2^f \ \cup \ \dot{J}_2 > 0$$

index ***f*** is for the maximum previous value of the corresponding stress invariant.

$$\dot{p} = K_t\,\dot{\varepsilon}_{kk} \qquad \dot{s}_{ij} = 2G_t\dot{e}_{ij}\,; \qquad 3\,\dot{p} = \dot{\sigma}_{kk}$$

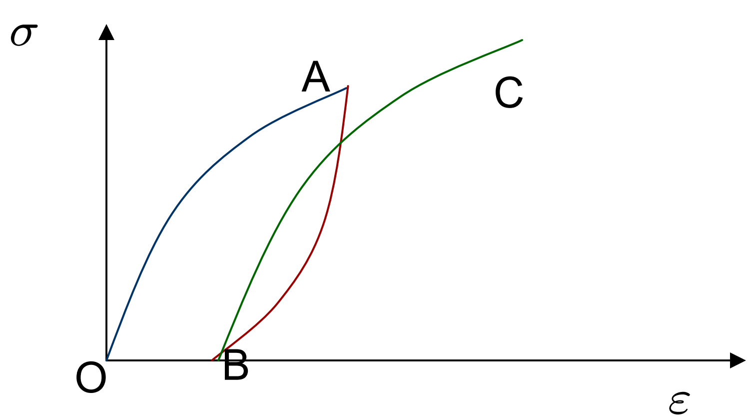
loading – unloading – reloading conditions :

$$K_t = K_{tL} \quad \mapsto \quad p = p^f \;\cup\; \dot{p} > 0 \qquad G_t = G_{tL} \quad \mapsto \quad J_2 = J_2^f \;\cup\; \dot{J}_2 > 0$$

$$K_t = K_{tU} \quad \mapsto \quad p \le p^f \;\cup\; \dot{p} < 0 \qquad G_t = G_{tU} \quad \mapsto \quad J_2 \le J_2^f \;\cup\; \dot{J}_2 < 0$$

$$K_t = K_{tR} \quad \mapsto \quad p < p^f \;\cup\; \dot{p} > 0 \qquad G_t = G_{tR} \quad \mapsto \quad J_2 < J_2^f \;\cup\; \dot{J}_2 > 0$$

index ***f*** is for the maximum previous value of the corresponding stress invariant.



uniaxial stress-strain relation
OA – loading;
AB – unloading;
BC – reloading.

Summary for Variable Modules Type Models

$\dot{p} = K_t \, \dot{\varepsilon}_{kk}$ $\dot{s}_{ij} = 2G_t \dot{e}_{ij} \,;$ $3\,\dot{p} = \dot{\sigma}_{kk}$ **+ L-U-R properties !**

1. These type of models are mainly based on *curve-fitting* techniques.
2. They represent a *special class of isotropic hypoelastic* materials with additional restriction – *incremental isotropic* form.
3. *No unique* stress – strain relation exists, since there are loading – unloading – reloading conditions.
4. As direct consequence of different material response coefficients the behaviour is *irreversible*.
5. The model may generate energy in load – unload cycles and therefore may be *thermodynamically inconsistent*.
6. *Computationally simple* and relatively *easy to fit to experimental data*.

! but often bias against paths different then used for the fit

At each stage of loading a typical structural material is characterized by a certain arrangement of dislocations, microcracks, voids and other flaws and defects.

Distributed deffects in materials and structures lead to:

- crack initiation and final failure;
- progressive material deterioration (decrease of strength).

→ material damage → Continuum damage mechanics

↓

use of internal state variables – damage variables

scalars, vectors, second and fourth order tensors

+

evolution equation (kinematic equation)
for the damage parameter

$$\sigma_{ij} = \varphi_{ij}\left(\boldsymbol{D}, \boldsymbol{\varepsilon}\right)$$

$$\varepsilon_{ij} = \psi_{ij}\left(\boldsymbol{D}, \boldsymbol{\sigma}\right)$$

Cordebois & Sidoroff (1983); Lemaitre (1984) ⟶ effective state hypothesis:

$$\exists\ \tilde{\sigma}_{ij}\ ,\ \tilde{\varepsilon}_{ij}\ :\quad \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\tilde{\sigma}_{ij}\tilde{\varepsilon}_{ij}$$

Strain energy of the damaged material is equivalent to the strain energy of the undamaged (effective) material

$$\Downarrow$$

effective stress: $$\tilde{\sigma}_{ij} = M_{ijkl}\left(\boldsymbol{D}\right)\,\sigma_{kl}$$

$M_{ijkl}\left(\boldsymbol{D}\right)$ - damage effective tensor

$\boldsymbol{D}$ - damage variable tensor

$$\Downarrow$$

$$\tilde{\sigma}_{ij} = \tilde{\varphi}_{ij}\left(\boldsymbol{\varepsilon}\right)\ ;\quad \tilde{\sigma}_{ij} = \varphi_{ij}\left(\tilde{\boldsymbol{\varepsilon}}\right)$$

$$\varepsilon_{ij} = \tilde{\psi}_{ij}\left(\boldsymbol{\sigma}\right)\ ;\quad \tilde{\varepsilon}_{ij} = \psi_{ij}\left(\tilde{\boldsymbol{\sigma}}\right)$$

\+

$$\dot{\boldsymbol{D}} = \dot{\boldsymbol{D}}\left(\tilde{\boldsymbol{\sigma}}\ ,\ \varpi\right)$$

evolutionary (kinematic) or damage growth equation

Example:

$$
\begin{pmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{33} \\ \tilde{\sigma}_{12} \\ \tilde{\sigma}_{23} \\ \tilde{\sigma}_{31} \end{pmatrix}
=
\underbrace{\begin{pmatrix}
\frac{1}{1-D_1} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{1}{1-D_2} & 0 & 0 & 0 & 0 \\
 & & \frac{1}{1-D_3} & 0 & 0 & 0 \\
 & & & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} & 0 & 0 \\
 & \text{symmetric} & & & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & 0 \\
 & & & & & \frac{1}{\sqrt{(1-D_3)(1-D_1)}}
\end{pmatrix}}_{M_{ijkl}(\boldsymbol{D})}
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}
$$

Anisotropic damage: damage variables D_1 , D_2 , D_3

characterize the deterioration along x_1 , x_2 , x_3

*in damage principle axes

Example:

Cordebois & Sidoroff (1983); Lemaitre (1984) $\longrightarrow$ effective state hypothesis:

$$\Downarrow$$

The complemetary elastic energy for a damage material is the same form as that of the undamaged material.

$$\Downarrow$$

$$\Phi\left(\boldsymbol{\sigma},\,\boldsymbol{D}\right)=\Phi^{e}\left(\tilde{\boldsymbol{\sigma}}\right)=\frac{1}{2}\,\tilde{\sigma}_{ij}\;C^{-1}_{ijkl}\;\tilde{\sigma}_{kl}=$$

linear isotropic elastic material

$$=\frac{1}{2}\,\sigma_{mn}\,\underbrace{M_{mnij}\;C^{-1}_{ijkl}\;M_{klrs}}_{\tilde{C}^{-1}_{ijkl}}\sigma_{rs}$$

Example:

$$\tilde{C}^{-1}_{ijkl}(D_1\,,\,D_2\,,\,D_3) =$$

$$\begin{pmatrix}
\frac{1}{E(1-D_1)^2} & \frac{-\nu}{E(1-D_1)(1-D_2)} & \frac{-\nu}{E(1-D_1)(1-D_3)} & 0 & 0 & 0 \\
 & \frac{1}{E(1-D_2)^2} & \frac{-\nu}{E(1-D_2)(1-D_3)} & 0 & 0 & 0 \\
 & & \frac{1}{E(1-D_3)^2} & 0 & 0 & 0 \\
 & & & \frac{2(1+\nu)}{E(1-D_2)(1-D_1)} & 0 & 0 \\
 & \text{symmetric} & & & \frac{2(1+\nu)}{E(1-D_3)(1-D_2)} & 0 \\
 & & & & & \frac{2(1+\nu)}{E(1-D_3)(1-D_1)}
\end{pmatrix}$$

anisotropic compliance matrix

Example:

Damage evolution (growth) equation:

$$dD_{ij} = d\mu \frac{\partial G}{\partial \tilde{\tau}} \frac{\partial \tilde{\tau}}{\partial \varepsilon_{ij}}$$

• damage potential function:

$$G\left(\tilde{\tau},\, \beta_D\right) = g\left(\tilde{\tau}\right) - h_D\left(\beta_D\right)$$

• strain-based energy norm:

$$\tilde{\tau} = \sqrt{\varepsilon_{kl}\, \tilde{C}_{klij}\, \varepsilon_{ij}}$$

• hardening function and hardening norm:

$$h_D(\beta_D)\,; \quad \beta_D = \sqrt{D_1^2 + D_2^2 + D_3^2}$$

• consistancy condition for the damage potential:

$$dG = \frac{\partial G}{\partial \tilde{\tau}} d\tilde{\tau} + \frac{\partial G}{\partial \beta_D} d\beta_D = 0$$

$$d\mu = m_{ij} d\varepsilon_{ij}$$

$$dD_{ij} = N_{ijkl}\, d\varepsilon_{kl}$$

Experimental observation: solid bodies expand when heated and shrink if cooled.

Duhamel – Neumann relation (experimental fact) states that elastic and thermal deformations are additive and for linear elastic material it reads:

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha_{kl}\Delta T\right)$$

C_{ijkl} are determined at $T=T_0$

If $\Delta T = T - T_0$ is not small, then $C_{ijkl}(T)$ and $\alpha_{kl}(T)$

There is one more unknown in thermoelasticity and this is temeprature ***T***

→ add a thermodynamic law for external thermodynamic process parameter *T*:

The phenomenological Fourier law for thermal conductivity or the Heat Equation:

$$q_i = \lambda_{ij}\, T_{,j}$$

q_i - heat flux vector [J·m−2·s−1]

λ_{ij} - thermal conductivity tensor [W·m−1·K−1]

$$\varepsilon_{ij} = \varepsilon_{ij}\left(\sigma_{kl}, T\right) \qquad \sigma_{ij} = \sigma_{ij}\left(\varepsilon_{kl}, T\right)$$

First and second thermodynamical laws read:

$$dU = \sigma_{ij} d\varepsilon_{ij} + dq^e$$ = mechanical work + heat supply

$$T ds = dq^e$$

T and *s* are absolute temperature and entropy,

q^e - heat flux per unit mass, *U* – internal energy

$$\Downarrow$$

Gibb's identity:

$$dU = \sigma_{ij} d\varepsilon_{ij} + T\,ds$$

$U\left(\varepsilon_{ij}, s\right)$ is a thermodynamical potential ⟷ ε_{ij}, s state parameters

(exact differential)

$U\left(\varepsilon_{ij}, s\right)$ thermodynamical potential ⟷ ε_{ij}, s state parameters

Constitutive equations:

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$$T = \frac{\partial U}{\partial s}$$

$F\left(\varepsilon_{ij}, T\right)$ thermodynamical potential ⟷ ε_{ij}, T state parameters

$F = U - s\,T$ - free energy ⟷ $dF = \sigma_{ij} d\varepsilon_{ij} - s\,dT$

(exact differential)

Constitutive equations:

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$$s = -\frac{\partial F}{\partial T}$$

Gibb‘s potential

$$G = F - \sigma_{ij}\,\varepsilon_{ij}$$

$G(\sigma_{ij}\,,\,T)$ thermodynamical potential ⟷ σ_{ij}, T state parameters

$$dG = -\,\varepsilon_{ij} d\sigma_{ij} - s\,dT$$

(exact differential)

Constitutive equations:

$$\varepsilon_{ij} = -\frac{\partial G}{\partial \sigma_{ij}} = \frac{\partial \Phi}{\partial \sigma_{ij}}$$

$$s = -\frac{\partial G}{\partial T}$$

$G(\sigma_{ij}\,,\,T)$ $\quad F\left(\varepsilon_{ij}, T\right)$ $\quad U\left(\varepsilon_{ij}, s\right)$

$G = F - \sigma_{ij}\,\varepsilon_{ij}$ $\quad F = U - s\,T$

Expanding free energy in a power series in the visinity of the undeformed state:

$$\varepsilon_{ij} = 0 \ , \ T = T_0$$

$$F\left(\varepsilon_{ij}, T\right) = F\left(0, T_0\right) + \frac{\partial F\left(0, T_0\right)}{\partial \varepsilon_{ij}} \varepsilon_{ij} + \frac{\partial F\left(0, T_0\right)}{\partial T} \left(T - T_0\right) +$$

0

$$0 = -s(0, T_0)$$

entropy for the undeformed state is 0

$$+ \frac{1}{2} \frac{\partial^2 F\left(0, T_0\right)}{\partial \varepsilon_{ij} \, \partial \varepsilon_{kl}} \varepsilon_{ij} \, \varepsilon_{kl} + \frac{\partial^2 F\left(0, T_0\right)}{\partial \varepsilon_{ij} \, \partial T} \varepsilon_{ij} \left(T - T_0\right) +$$

$$+ \frac{1}{2} \frac{\partial^2 F\left(0, T_0\right)}{\partial T^2} \left(T - T_0\right)^2 + \ldots$$

$$\Downarrow$$

$$\sigma_{ij}\left(\varepsilon_{ij}, T\right) = \frac{\partial F}{\partial \varepsilon_{ij}} = \frac{\partial F\left(0, T_0\right)}{\partial \varepsilon_{ij}} + \frac{\partial^2 F\left(0, T_0\right)}{\partial \varepsilon_{ij} \, \varepsilon_{kl}} \varepsilon_{kl} + \frac{\partial^2 F\left(0, T_0\right)}{\partial \varepsilon_{ij} \, \partial T} \left(T - T_0\right)$$

0

$$\sigma_{ij} = 0 \ , \ T = T_0 \ \Rightarrow \ \varepsilon_{ij} = 0$$

$$\sigma_{ij}\left(\varepsilon_{ij},T\right)=\frac{\partial F}{\partial \varepsilon_{ij}}=\boxed{\frac{\partial^2 F\,(0,T_0)}{\partial \varepsilon_{ij}\,\varepsilon_{kl}}}\,\varepsilon_{kl}+\boxed{\frac{\partial^2 F\,(0,T_0)}{\partial \varepsilon_{ij}\,\partial T}}\,(T-T_0)$$

$$C^T_{ijkl} \qquad\qquad -\alpha_{ij}$$

$$\Downarrow$$

$$F\left(\varepsilon_{ij},T\right)=\frac{1}{2}C^T_{ijkl}\;\varepsilon_{ij}\,\varepsilon_{kl}-\alpha_{ij}\,\varepsilon_{ij}\,(T-T_0)+F_0(T,T_0)$$

Constitutive equations:

$$\sigma_{ij}=C^T_{ijkl}\;\varepsilon_{kl}\;-\;\alpha_{ij}\;(T-T_0)$$ *Duhamel – Neumann* relation

$$s=-\frac{\partial F}{\partial T}=\alpha_{ij}\,\varepsilon_{ij}-\frac{\partial F_0}{\partial T}$$

$$-\frac{\partial F_0}{\partial T}=c_\varepsilon\ \ln\left(1+\frac{T-T_0}{T_0}\right)$$ with $$c_\varepsilon=-T\,\frac{\partial^2 F_0}{\partial T^2}$$

$$\sigma_{ij} = C^{T}_{ijkl}\ \varepsilon_{kl} - \alpha_{ij}\ (T - T_0)$$

$$\varepsilon_{ij} = 0 \ \Rightarrow\ \sigma_{ij} = \underbrace{\alpha_{ij}\ (T - T_0)}_{\text{thermal stresses}}$$

C^{T}_{ijkl} are to be defined from isothermal tests (constant temperature)

$$\sigma_{ij} = C^{s}_{ijkl}\ \varepsilon_{kl} - \gamma_{ij}\, s$$

C^{s}_{ijkl} are to be defined from adiabatic tests (constant entropy)

$$\varepsilon_{ij} = D^{T}_{ijkl}\ \sigma_{kl} + \beta_{ij}\ (T - T_0)$$

$$\sigma_{ij} = 0 \ \Rightarrow\ \varepsilon_{ij} = \underbrace{\beta_{ij}\ (T - T_0)}_{\text{thermal expansion}}$$

$$\sigma_{ij} = C^T_{ijkl}\ \varepsilon_{kl} - \alpha_{ij}\ (T - T_0)$$

$\Downarrow$ linear isotropic material

$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\,\mu\,\varepsilon_{ij} - (3\lambda + 2\mu)\alpha_T\ (T - T_0)\ \delta_{ij}$$

linear isotropic thermoelastic constitutivr law