isotropic nonlinear Green (hyper) elastic material

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{kl})}{\partial \varepsilon_{ij}} \qquad \varepsilon_{ij} = \frac{\partial \Phi(\sigma_{kl})}{\partial \sigma_{ij}} \qquad \text{(EN3)}$$

If the material is isotropic $\Rightarrow$

$$W(\varepsilon_{kl}) = W\left(I_1^\varepsilon, I_2^\varepsilon, I_3^\varepsilon\right)$$

e.g. a polynomial function

From (EN3) $\Rightarrow$

$$\sigma_{ij} = \frac{\partial W}{\partial I_1^\varepsilon}\frac{\partial I_1^\varepsilon}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial I_2^\varepsilon}\frac{\partial I_2^\varepsilon}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial I_3^\varepsilon}\frac{\partial I_3^\varepsilon}{\partial \varepsilon_{ij}}$$

$$\Downarrow$$

$$\sigma_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \varepsilon_{ij} + \alpha_3 \varepsilon_{ik}\varepsilon_{kj} \qquad \alpha_i = \alpha_i\left(I_j^\varepsilon\right) = \frac{\partial W}{\partial I_i^\varepsilon}$$

as

$$\frac{\partial^2 W}{\partial I_i^\varepsilon \, \partial I_j^\varepsilon} = \frac{\partial^2 W}{\partial I_j^\varepsilon \, \partial I_i^\varepsilon} \quad \Rightarrow \quad \boxed{\frac{\partial \alpha_i}{\partial I_j^\varepsilon} = \frac{\partial \alpha_j}{\partial I_i^\varepsilon}}$$

! restrictions on the nonlinear elastic moduli

e.g. polynomial functions

Remark on strain energy and complementary energy densities

W Φ

•Definition for homogeneous of order *n* function:

A function which satisfies:

$$f\left(tx_1, tx_2 \dots tx_k\right) = t^n f\left(x_1, x_2 \dots x_k\right)$$

for a fixed *n* .

•Euler's homogeneous function theorem states that:

$$x_i \frac{\partial f}{\partial x_i} = n\, f(\boldsymbol{x})$$

For *linear isotropic elastic material* it reads:

$$\sigma_{ij} = K \; \varepsilon_{kk}\delta_{ij} + 2G \; e_{ij} = \frac{\partial W}{\partial I_1^{\varepsilon}}\delta_{ij} + \frac{\partial W}{\partial J_2^{\varepsilon}} e_{ij} \quad \Rightarrow \quad W\left(I_1^{\varepsilon},\, J_2^{\varepsilon}\right) = \frac{K}{2}\left(I_1^{\varepsilon}\right)^2 + 2G\, J_2^{\varepsilon}$$

homogeneous of order 2

$$\Rightarrow \quad \sigma_{ij}\,\varepsilon_{ij} = \varepsilon_{ij}\frac{\partial W}{\partial \varepsilon_{ij}} = 2\,W$$

$$\Rightarrow \quad W = \Phi$$

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$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \; ; \qquad \Phi\left(\sigma_{kl}\right) = \Phi\left(I_1 \, , \, J_2 \, , \, J_3\right)$$

complementary energy density

$$I_1 = \sigma_{kk} \qquad J_2 = \tfrac{1}{2} s_{mn} s_{mn} \qquad J_3 = \tfrac{1}{3} s_{mn} s_{nk} s_{km}$$

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}}$$

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij} \; ; \qquad \frac{\partial J_2}{\partial \sigma_{ij}} = \frac{\partial J_2}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial \sigma_{ij}} = s_{mn}\left(\delta_{im}\,\delta_{jn} - \tfrac{1}{3}\,\delta_{mn}\delta_{ij}\right) = s_{ij}$$

$$\frac{\partial J_3}{\partial \sigma_{ij}} = \frac{\partial J_3}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial \sigma_{ij}} = s_{nk}\, s_{km}\left(\delta_{im}\,\delta_{jn} - \tfrac{1}{3}\,\delta_{mn}\delta_{ij}\right) = s_{jk}\, s_{ki} - \tfrac{2}{3} J_2 \delta_{ij}$$

! *** Units: $\Phi \; [Pa]$

isotropic nonlinear Green (hyper) elastic material

Example 3**

$$\Phi\left(\sigma_{kl}\right) = \Phi\left(I_1\,,\,J_2\,,\,J_3\right) = a\,I_1^2 + b\,J_2 + c\,J_3^{2/3}$$

$$\varepsilon_{ij} = \frac{\partial\Phi}{\partial I_1}\,\delta_{ij} + \frac{\partial\Phi}{\partial J_2}\,s_{ij} + \frac{\partial\Phi}{\partial J_3}\left(s_{jk}\,s_{ki} - \frac{2}{3}\,J_2\delta_{ij}\right)$$

$$\frac{\partial\Phi}{\partial I_1} = 2\,a\,I_1 \qquad \frac{\partial\Phi}{\partial J_2} = b \qquad \frac{\partial\Phi}{\partial J_3} = \frac{2}{3}\,c\,J_3^{-1/3}$$

$$\Downarrow$$

$$\varepsilon_{ij} = \alpha_1^{\varepsilon}\,\delta_{ij} + \alpha_2^{\varepsilon} s_{ij} + \alpha_3^{\varepsilon} s_{ik} s_{kj}$$

$$\alpha_1^{\varepsilon} = 2\,a\,I_1 - \frac{4}{9}\,c\frac{J_2}{J_3^{1/3}} \qquad \alpha_2^{\varepsilon} = b \qquad \alpha_3^{\varepsilon} = \frac{2}{3}\,c\frac{J_2}{J_3^{1/3}}$$

! ** Units: $\Phi\ [Pa] \ \Rightarrow\ a,\ b,\ c\ [Pa^{-1}]$

isotropic elastic material

$$\dot{\sigma}_{ij} = F_{ij}\left(\dot{\varepsilon}_{kl},\, \sigma_{kl}\right)$$

Hypoelastic constitutive model

F_{ij} → elastic response function

For isotropic *time-independent* material behaviour the strain-stress relationship can be written in incrementally linear form:

$$d\sigma_{ij} = C_{ijkl}(\sigma_{kl})\, d\varepsilon_{kl} \qquad \text{(Hypo1)}$$

$C_{ijkl}(\sigma_{kl})$ → tangential stiffness tensor

$$d\varepsilon_{ij} = D_{ijkl}(\varepsilon_{kl})\, d\sigma_{kl} \qquad \text{(Hypo2)}$$

$D_{ijkl}(\varepsilon_{kl})$ → tangential compliance tensor

$d\varepsilon_{ij}$, $d\sigma_{kl}$ → Strain and stress increment tensors

* mostly used for integration of elastoplastic problems

$$p = K\,\varepsilon_{kk} \qquad s_{ij} = 2Ge_{ij}\,; \qquad 3\,p = \sigma_{kk}$$

isotropic linear elastic material

stress inaviants:

$$p = \frac{1}{3}\sigma_{kk}\,; \qquad q = \sqrt{\frac{3}{2}\,s_{ij}s_{ij}} = \sqrt{3\,J_2}$$

corresponding strain inaviants:

$$\theta = \varepsilon_{kk}\,; \quad \epsilon = \sqrt{\frac{2}{3}\,e_{ij}e_{ij}} = 2\sqrt{\frac{J_2^{\varepsilon}}{3}}\,; \quad e = 2\sqrt{3\,J_2^{\varepsilon}}$$

$$s_{ij} = 2Ge_{ij} \quad \Rightarrow \quad s_{ij}s_{ij} = 4G^2 e_{ij}e_{ij} \quad \Rightarrow \quad q = 3G\epsilon = G\,e$$

Incremental form:

$$dp = K\,d\theta \qquad dq = 3G\,d\epsilon = G\,de$$

$$\dot{p} = K\,\dot{\theta} \qquad \dot{q} = 3G\dot{\epsilon} = G\,\dot{e}$$

Incremental form:

isotropic linear elastic material

$$\dot{p} = K\,\dot{\theta} \qquad \dot{q} = 3G\dot{\epsilon} = G\,\dot{e}$$

Back to the relation between increments of stress and strain components:

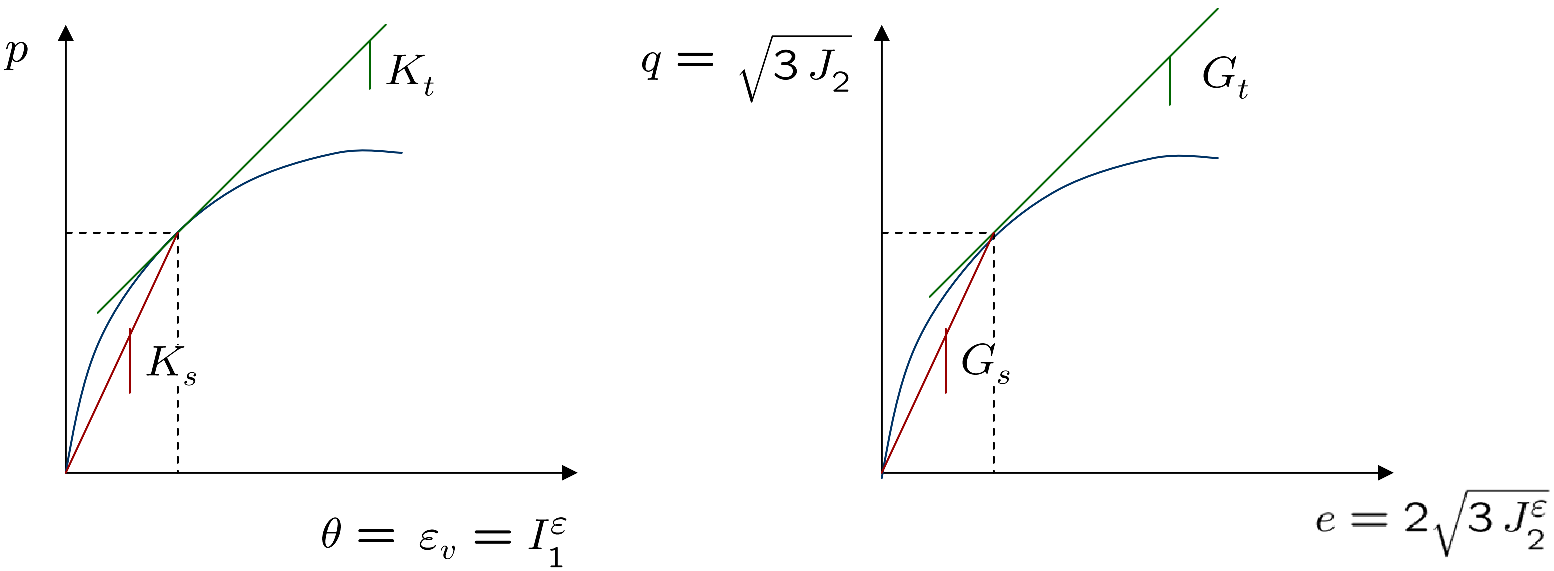
$$\dot{\sigma}_{ij} = \underbrace{\left[2G\delta_{ik}\delta_{jl} + \left(K - \frac{2}{3}G\right)\delta_{kl}\delta_{ij}\right]}_{C^t_{ijkl}}\dot{\varepsilon}_{kl}$$

$$\begin{pmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{33} \\ \dot{\sigma}_{12} \\ \dot{\sigma}_{23} \\ \dot{\sigma}_{31} \end{pmatrix} = \begin{pmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ & & K+\frac{4}{3}G & 0 & 0 & 0 \\ & \text{symmetric} & & G & 0 & 0 \\ & & & & G & 0 \\ & & & & & G \end{pmatrix} \begin{pmatrix} \dot{\varepsilon}_{11} \\ \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{33} \\ 2\dot{\varepsilon}_{12} \\ 2\dot{\varepsilon}_{23} \\ 2\dot{\varepsilon}_{31} \end{pmatrix}$$

Example 4: an incremental relation based on secant moduli

$$p = K_s \varepsilon_{kk} \qquad s_{ij} = 2G_s e_{ij}\,; \qquad 3p = \sigma_{kk}$$

$$K_s = K_s\left(I_1^{\varepsilon}\right) \qquad G_s = G_s\left(J_2^{\varepsilon}\right) \qquad \theta = \varepsilon_{kk}\,;$$



$$dp = \left(K_s + \frac{dK_s}{d\theta}\theta\right) d\theta\,; \qquad \dot{q} = \left(G_s + e\,\frac{dG_s}{de}\right)\dot{e}$$

Example 4: an incremental relation based on secant moduli (nonlinear elasticity)

$$p = K_s \varepsilon_{kk} \qquad s_{ij} = 2G_s e_{ij} \, ; \qquad 3\,p = \sigma_{kk}$$

$$p = K_s(\theta)\theta \;\rightarrow\; dp = K_s(\theta)d\theta + \underbrace{\frac{dK_s}{d\theta}\theta\, d\theta}_{\theta\, dK_s} \;\rightarrow\; dp = \underbrace{\left(K_s + \frac{dK_s}{d\theta}\theta\right)}_{K_t} d\theta$$

$$s_{ij} = 2G_s(\epsilon)e_{ij} \;\rightarrow\; s_{ij}s_{ij} = 4G_s^2(\epsilon)e_{ij}e_{ij} \;\rightarrow\; q^2 = 9G_s^2(\epsilon)\epsilon^2 \;\rightarrow$$

$$q = 3G_s(\epsilon)\epsilon \;\rightarrow\; dq = 3G_s(\epsilon)d\epsilon + 3\frac{dG_s}{d\epsilon}\epsilon\, d\epsilon \;\rightarrow\; dq = 3\underbrace{\left(G_s + \frac{dG_s}{d\epsilon}\epsilon\right)}_{G_t} d\epsilon$$

Example 4: an incremental relation based on secant moduli (nonlinear elasticity)

$$\dot{p} = \left(K_s + \theta \frac{dK_s}{d\theta}\right)\dot{\theta} \; ; \qquad \dot{q} = \left(G_s + e\frac{dG_s}{de}\right)\dot{e}$$

$$\dot{p} = K_t\,\dot{\theta} \; ; \qquad \dot{q} = G_t \dot{e}$$

$$\frac{dK_s}{d\theta} = \frac{K_t - K_s}{\theta} \qquad \frac{dG_s}{de} = \frac{G_t - G_s}{e}$$

Tangential stiffness matrix:

$$\begin{pmatrix} K_t & 0 \\ 0 & G_t \end{pmatrix}$$

$$\begin{pmatrix} dp \\ dq \end{pmatrix} = \begin{pmatrix} K_t & \boxed{0} \\ \boxed{0} & G_t \end{pmatrix}\begin{pmatrix} d\theta \\ de \end{pmatrix} \qquad \begin{pmatrix} dp \\ dq \end{pmatrix} = \begin{pmatrix} K_t & 0 \\ 0 & 3G_t \end{pmatrix}\begin{pmatrix} d\theta \\ d\epsilon \end{pmatrix}$$

Example 4: an incremental relation based on secant moduli (nonlinear elasticity)

$$\dot{p} = \left(K_s + \theta \frac{dK_s}{d\theta}\right) \dot{\theta} \; ; \qquad \dot{q} = \left(G_s + e \frac{dG_s}{de}\right) \dot{e}$$

relates the increments of invariants

$$\Downarrow$$

Back to the stress-strain tensors incremental relation:

$$\dot{\sigma}_{ij} = \underbrace{2\left[G_s \underbrace{\delta_{ik}\delta_{jl} + \left(\frac{K_t}{2} - \frac{G_s}{3}\right)\delta_{ij}\delta_{kl}}_{A_{ijkl}/2} + 6\underbrace{\frac{G_t - G_s}{e^2} e_{ij}e_{kl}}_{B_{ijkl}/2}\right]}_{C^t_{ijkl}} \dot{\varepsilon}_{kl}$$

tangential stiffness matrix

$$C^t_{ijkl} = A_{ijkl} + B_{ijkl}$$

Isotropic linear elastic incremental law

$$\dot{\sigma}_{ij} = \left[2G\delta_{ik}\delta_{jl} + \left(K - \frac{2}{3}G \right) \delta_{kl}\delta_{ij} \right] \dot{\varepsilon}_{kl}$$

$$\dot{\sigma}_{ij} = 2 \left[G_s\delta_{ik}\delta_{jl} + \left(\frac{K_t}{2} - \frac{G_s}{3} \right) \delta_{ij}\delta_{kl} + 6\, \frac{G_t - G_s}{e^2}\, e_{ij}e_{kl} \right] \dot{\varepsilon}_{kl}$$

Isotropic nonlinear elastic incremental law

$$\frac{dG_s}{de} = \frac{G_t - G_s}{e}$$

$$\begin{pmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{33} \\ \dot{\sigma}_{12} \\ \dot{\sigma}_{23} \\ \dot{\sigma}_{31} \end{pmatrix} = \begin{pmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ & & K+\frac{4}{3}G & 0 & 0 & 0 \\ & \text{symmetric} & & \underline{G} & 0 & 0 \\ & & & & \underline{G} & 0 \\ & & & & & \underline{G} \end{pmatrix} \begin{pmatrix} \dot{\varepsilon}_{11} \\ \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{33} \\ 2\dot{\varepsilon}_{12} \\ 2\dot{\varepsilon}_{23} \\ 2\dot{\varepsilon}_{31} \end{pmatrix}$$

Example 5: an incremental relation based on hyperelasticity

$$W = W(\theta, \epsilon)$$

strain energy density function

$$\Downarrow$$

$$p = \frac{\partial W}{\partial \theta} \qquad q = \frac{\partial W}{\partial \epsilon}$$

and

$$p = p(\theta, \epsilon) \qquad q = q(\theta, \epsilon)$$

$$\Downarrow$$

$$K_t = \frac{\partial p}{\partial \theta} = \frac{\partial^2 W}{\partial \theta^2} \qquad 3G_t = \frac{\partial q}{\partial \epsilon} = \frac{\partial^2 W}{\partial \epsilon^2} \qquad R = \frac{\partial p}{\partial \epsilon} = \frac{\partial q}{\partial \theta} = \frac{\partial^2 W}{\partial \epsilon \partial \theta}$$

$$\begin{pmatrix} dp \\ dq \end{pmatrix} = \begin{pmatrix} K_t & \boxed{R} \\ \boxed{R} & 3G_t \end{pmatrix} \begin{pmatrix} d\theta \\ d\epsilon \end{pmatrix}$$