Concrete, biological tissues, rocks, soils ...
exhibit nonlinear mechanical behavour even under small deformations

General tensor algebra

Cayley – Hamilton theorem: All positive integer powers of any second-order tensor t_{ij} can be expressed as linear combination of δ_{ij} , t_{ij} and $t_{ik}t_{kj}$ with coefficients that are polynomial functions of the three invariants of t_{ij} .

$$\delta_{ij},\ t_{ij},\ t_{ik}t_{kj} \rightarrow \boldsymbol{I},\ \boldsymbol{t},\ \boldsymbol{t}^2$$

$$\boldsymbol{t}^N = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{t} + \alpha_3 \boldsymbol{t}^2$$

$$\alpha_i = \alpha_i \left(I_1^t,\ I_2^t,\ I_3^t \right) \qquad i = 1, 2, 3$$

Cauchy elastic material

$$\sigma_{ij} = \varphi_{ij}\left(\varepsilon_{kl}\right) \qquad\qquad \varepsilon_{ij} = \varphi_{ij}^{-1}\left(\sigma_{kl}\right)$$

In case φ_{ij} is an analytical function of ε_{kl} and regarding *Cayley-Hamilton theorem* φ_{ij} can be expanded as:

$$\sigma_{ij} = \varphi_{ij}\left(\varepsilon_{kl}\right) = \alpha_1\,\delta_{ij} + \alpha_2\varepsilon_{ij} + \alpha_3\varepsilon_{ik}\varepsilon_{kj}$$

(EN1)

+ isotropy

$$\alpha_i = \alpha_i\left(I_1^\varepsilon,\, I_2^\varepsilon,\, I_3^\varepsilon\right) \qquad i = 1,2,3$$

And respectively:

$$\varepsilon_{ij} = \varphi_{ij}^{-1}\left(\sigma_{kl}\right) = \alpha_1^\varepsilon\,\delta_{ij} + \alpha_2^\varepsilon\sigma_{ij} + \alpha_3^\varepsilon\sigma_{ik}\sigma_{kj}$$

(EN2)

+ isotropy

$$\alpha_i^\varepsilon = \alpha_i^\varepsilon\left(I_1,\, I_2,\, I_3\right) \qquad i = 1,2,3$$

(EN1) and (EN2) are nonlinear strain-stress relations giving the constitutive laws for isotropic nonlinear Cauchy elastic material.

Prove: elastic, isotropic and reversible

Isotropic linear elastic material (Green type)

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\alpha_i^{\varepsilon} = \alpha_i^{\varepsilon}\left(I_1,\, I_2,\, I_3\right) \qquad i = 1,2,3$$

$$\Downarrow$$

$$\alpha_1^{\varepsilon} = \alpha_1^{\varepsilon}\left(I_1\right) = -\frac{\nu}{E}I_1 \qquad \alpha_2^{\varepsilon} = const = \frac{1+\nu}{E} \qquad \alpha_3^{\varepsilon} = 0$$

Used notations for the stress invariants:

$$\left.\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{11} + \sigma_{22} + \sigma_{33} \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}\right\}$$

The 3 independent stress invariants expressed in terms of principle stresses

Stress invariants:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

$$I_\sigma = \mathrm{tr}\,\boldsymbol{\sigma} = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 = I_1$$

$$II_\sigma = \mathrm{tr}\,\boldsymbol{\sigma}^2 = \sigma_{ij}\sigma_{ij} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = I_1^2 - 2\,I_2$$

$$III_\sigma = \mathrm{tr}\,\boldsymbol{\sigma}^3 = \sigma_{ik}\sigma_{kj}\sigma_{ji} = \sigma_1^3 + \sigma_2^3 + \sigma_3^3 = I_1^3 - 3\,I_1\,I_2 + 3I_3$$

$$II_s = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{2}\left(II_\sigma - \frac{1}{3}I_\sigma^2\right)$$

$$III_s = \frac{1}{3}s_{ij}s_{jk}s_{ki} = \frac{1}{3}\left(III_\sigma - I_\sigma II_\sigma - \frac{2}{9}I_\sigma^3\right)$$

Stress invariants:

$$II_\sigma^2 = \mathrm{tr}\,\boldsymbol{\sigma}^2 = I_1^2 - 2\,I_2$$

$$II_s^2 = \tfrac{1}{2} s_{ij} s_{ij} = \tfrac{1}{2}\left(II_\sigma^2 - \tfrac{1}{3} I_\sigma^2\right)$$

$$I_3 = \tfrac{1}{6}\left(I_\sigma^3 - 3\,I_\sigma II_\sigma^2 + 2\,III_\sigma\right)$$

$$III_\sigma = I_1^3 - 3I_1 I_2 + 3I_3$$

$$III_s = \tfrac{1}{3} s_{ij} s_{jk} s_{ki} = \tfrac{1}{3}\left(III_\sigma - I_\sigma II_\sigma^2 - \tfrac{2}{9} I_\sigma^3\right)$$

$$III_s = I_3 - \tfrac{1}{3} I_1 I_2 - \tfrac{1}{27} I_1^3$$

$$II_2^2 = \tfrac{1}{3} I_1^2 - I_2$$

$$I_3 = III_s + \tfrac{1}{27} I_\sigma^3 - \tfrac{1}{3} I_\sigma II_s^2$$

Isotropic linear elastic constitutive relationship

Elastic parameters:

$$E = 2G(\nu + 1) \quad E = \frac{9KG}{3K+G} \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad \nu = \frac{3K-2G}{6K+2G}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)}$$

Young's modulus (*E*); bulk modulus (*K*); shear modulus (*G,μ*); Poisson's ratio (*v*)

[Pa]	**[Pa]**	**[Pa]**	**[-]**

Isotropic linear elastic material (Green)

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\alpha_i^{\varepsilon} = \alpha_i^{\varepsilon}(I_1, I_2, I_3) \qquad i = 1,2,3$$

$$\Downarrow$$

$$\alpha_1^{\varepsilon} = \alpha_1^{\varepsilon}(I_1) = -\frac{\nu}{E}I_1 \qquad \alpha_2^{\varepsilon} = const = \frac{1+\nu}{E} \qquad \alpha_3^{\varepsilon} = 0$$

Isotropic nonlinear Cauchy elastic material

1/E 1/E

$$\varepsilon_{ij} = (1+\nu)F(I_1, J_2, J_3)\sigma_{ij} - \nu F(I_1, J_2, J_3)\sigma_{kk}\delta_{ij}$$

$$\alpha_1^{\varepsilon} = \alpha_1^{\varepsilon}(I_1\, J_2, J_3) = -\nu F(I_1, J_2, J_3) \qquad \alpha_3^{\varepsilon} = 0$$

$$\alpha_2^{\varepsilon} = \alpha_2^{\varepsilon}(I_1\, J_2, J_3) = (1+\nu) F(I_1, J_2, J_3)$$

Isotropic nonlinear Cauchy elastic material

Example 1* (for the generalization of the isotropic linear elastic stress-strain relations)

$$\varepsilon_{ij} = (1+\nu) F(I_1,\, J_2,\, J_3)\, \sigma_{ij} - \nu\, F(I_1,\, J_2,\, J_3)\, \sigma_{kk} \delta_{ij}$$

$$\alpha_1^{\varepsilon} = \alpha_1^{\varepsilon}\,(I_1\, J_2,\, J_3) = -\nu\, F(I_1,\, J_2,\, J_3)\, I_1$$

$$\alpha_2^{\varepsilon} = \alpha_2^{\varepsilon}\,(I_1\, J_2,\, J_3) = (1+\nu)\, F(I_1,\, J_2,\, J_3) \qquad \alpha_3^{\varepsilon} = 0$$

⟶ degenerates to linear elastic if $F(I_1,\, J_2,\, J_3) = const = \frac{1}{E}$

Used notations for the stress invariants:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{kk}$$

1st stress invariant

$$J_2 = \frac{1}{2} s_{ij} s_{ij}$$

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}$$

2nd and 3rd stress deviator invariants.

Note: 1st stress deviator invariant is 0 : $J_1 = s_{ij}\delta_{ij} = 0$

(Any deviator tensor is traceless)

Isotropic nonlinear Cauchy elastic material

Example 2* (for the generalization of the isotropic linear elastic stress-strain relations)

$$\sigma_{ij} = K_s \varepsilon_{kk}\delta_{ij} + 2G_s e_{ij} \qquad K_s = K_s\left(I_1^\varepsilon,\, J_2^\varepsilon,\, J_3^\varepsilon\right) \qquad \text{(Ex2.1)}$$

$$G_s = G_s\left(I_1^\varepsilon,\, J_2^\varepsilon,\, J_3^\varepsilon\right)$$

The elastic bulk and shear moduli are taken as scalar functions of the stress/ or strain invariants. K_s and G_s are called *secant bulk modulus* and *secant shear modulus*.

(Ex2.1) $\Rightarrow$ $p = K_s \varepsilon_{kk}$ $\qquad s_{ij} = 2G_s e_{ij}\,;$ $\qquad 3\,p = \sigma_{kk}$ (Ex2.2)

hydrostatic *deviatoric*

decomposition

Corresponding stress invariants are used according to the decomposition of the stress-strain relation in (Ex2.2) to hydrostatic and deviatoric relations. Notations are:

$$I_1^\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$ 1st strain invariant

$$J_2^\varepsilon = \frac{1}{2} e_{ij} e_{ij} \qquad J_3^\varepsilon = \frac{1}{3} e_{ij} e_{jk} e_{ki}$$ 2nd and 3rd strain deviator invariants

* concrete, granular matrials

Isotropic nonlinear Cauchy elastic material

Example 2* (continuation)

In general the secant moduli may be arbitrary functions of strain (strain deviator) invariants.

$$K_s = K_s\left(I_1^{\varepsilon},\, J_2^{\varepsilon},\, J_3^{\varepsilon}\right) \qquad G_s = G_s\left(I_1^{\varepsilon},\, J_2^{\varepsilon},\, J_3^{\varepsilon}\right) \qquad \text{(E2Mod1)}$$

Strain energy path independency leads to:

$$W\left(\varepsilon_{ij}\right) = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \cdots = \int_0^{J_2^{\varepsilon}} 2G_s dJ_2^{\varepsilon} + \int_0^{I_1^{\varepsilon}} \frac{1}{2} K_s d(I_1^{\varepsilon})^2$$

$$\Downarrow$$

! restrictions on the nonlinear elastic moduli (E2Mod1)

$$G_s = G_s\left(J_2^{\varepsilon}\right) \qquad K_s = K_s\left(I_1^{\varepsilon}\right)$$

→ an example of admissible with energy path independency secant moduli

* concrete, granular matrials

Summary for Cauchy elastic material

$$\sigma_{ij} = \varphi_{ij}\left(\varepsilon_{kl}\right) = \alpha_1\,\delta_{ij} + \alpha_2\varepsilon_{ij} + \alpha_3\varepsilon_{ik}\varepsilon_{kj} \qquad \varepsilon_{ij} = \varphi_{ij}^{-1}\left(\sigma_{kl}\right)$$

1. Stresses and strains are *reversible* and *path independent*.
2. Mostly the models are formulated by simple modification of the isotropic linear elasticity relations based on variable *secant moduli*.
3. Often the material parameters have well-defined physical relations to the observed stress-strain behavior and are *easily determined* from experimental data.
4. The material *secant stiffness* and *compliance* matrices are generally *symmetric*.
5. *Thermodynamic laws may be violated* since the model may generate energy for some load-unload stress paths (*reversibility and path independency of energy function are not guaranteed*).

isotropic nonlinear Green (hyper) elastic material

$$\sigma_{ij} = \frac{\partial W\left(\varepsilon_{kl}\right)}{\partial \varepsilon_{ij}} \qquad \varepsilon_{ij} = \frac{\partial \Phi\left(\sigma_{kl}\right)}{\partial \sigma_{ij}} \qquad \text{(EN3)}$$

If the material is isotropic $\Rightarrow$

$$W\left(\varepsilon_{kl}\right) = W\left(I_1^{\varepsilon}, I_2^{\varepsilon}, I_3^{\varepsilon}\right)$$

polynomial function

From (EN3) $\Rightarrow$

$$\sigma_{ij} = \frac{\partial W}{\partial I_1^{\varepsilon}} \frac{\partial I_1^{\varepsilon}}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial I_2^{\varepsilon}} \frac{\partial I_2^{\varepsilon}}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial I_3^{\varepsilon}} \frac{\partial I_3^{\varepsilon}}{\partial \varepsilon_{ij}}$$

$\Downarrow$

$$\sigma_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \varepsilon_{ij} + \alpha_3 \varepsilon_{ik} \varepsilon_{kj} \qquad \alpha_i = \alpha_i\left(I_j^{\varepsilon}\right) = \frac{\partial W}{\partial I_i^{\varepsilon}}$$

as

$$\frac{\partial^2 W}{\partial I_i^{\varepsilon}\, \partial I_j^{\varepsilon}} = \frac{\partial^2 W}{\partial I_j^{\varepsilon}\, \partial I_i^{\varepsilon}} \qquad \Rightarrow \qquad \boxed{\frac{\partial \alpha_i}{\partial I_j^{\varepsilon}} = \frac{\partial \alpha_j}{\partial I_i^{\varepsilon}}}$$

! restrictions to the nonlinear elastic moduli

polynomial functions

isotropic nonlinear Green (hyper) elastic material

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \; ; \qquad \Phi\left(\sigma_{kl}\right) = \Phi\left(I_1\,,\, J_2\,,\, J_3\right)$$

complementary energy density

$$I_1 = \sigma_{kk} \qquad J_2 = \tfrac{1}{2}\, s_{mn} s_{mn} \qquad J_3 = \tfrac{1}{3}\, s_{mn} s_{nk} s_{km}$$

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}}$$

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij} \; ; \qquad \frac{\partial J_2}{\partial \sigma_{ij}} = \frac{\partial J_2}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial \sigma_{ij}} = s_{mn}\left(\delta_{im}\,\delta_{jn} - \tfrac{1}{3}\,\delta_{mn}\delta_{ij}\right) = s_{ij}$$

$$\frac{\partial J_3}{\partial \sigma_{ij}} = \frac{\partial J_3}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial \sigma_{ij}} = s_{nk}\, s_{km}\left(\delta_{im}\,\delta_{jn} - \tfrac{1}{3}\,\delta_{mn}\delta_{ij}\right) = s_{jk}\, s_{ki} - \tfrac{2}{3}\, J_2 \delta_{ij}$$

! *** Units: $\Phi \; [Pa]$

isotropic nonlinear Green (hyper) elastic material

Example 3**

$$\Phi\left(\sigma_{kl}\right) = \Phi\left(I_1\,,\; J_2\,,\; J_3\right) = a\,I_1^2 + b\,J_2 + c\,J_3^{2/3}$$

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial I_1}\,\delta_{ij} + \frac{\partial \Phi}{\partial J_2}\,s_{ij} + \frac{\partial \Phi}{\partial J_3}\left(s_{jk}\,s_{ki} - \frac{2}{3}\,J_2\delta_{ij}\right)$$

$$\frac{\partial \Phi}{\partial I_1} = 2\,a\,I_1 \qquad \frac{\partial \Phi}{\partial J_2} = b \qquad \frac{\partial \Phi}{\partial J_3} = \frac{2}{3}\,c\,J_3^{-1/3}$$

$$\Downarrow$$

$$\varepsilon_{ij} = \alpha_1^{\varepsilon}\,\delta_{ij} + \alpha_2^{\varepsilon} s_{ij} + \alpha_3^{\varepsilon} s_{ik} s_{kj}$$

$$\alpha_1^{\varepsilon} = 2\,a\,I_1 - \frac{4}{9}\,c\frac{J_2}{J_3^{1/3}} \qquad \alpha_2^{\varepsilon} = b \qquad \alpha_3^{\varepsilon} = \frac{2}{3}\,c\frac{J_2}{J_3^{1/3}}$$

! ** Units: $\Phi\;[Pa] \;\Rightarrow\; a,\; b,\; c\;[Pa^{-1}]$

Summary for Green elastic material
Hyper-elastic material

$$\sigma_{ij} = \frac{\partial W\left(\varepsilon_{kl}\right)}{\partial \varepsilon_{ij}} \qquad \varepsilon_{ij} = \frac{\partial \Phi\left(\sigma_{kl}\right)}{\partial \sigma_{ij}}$$

1. Stresses and strains are *reversible* and *path independent*.
2. Constitutive laws based on assumed functions W and Φ have great *mathematical* capabilities and different *general* relations can be derived.
3. The material constants involved have *no direct* physical interpretation and may require *complicated testing programs* for model identification.
4. Material *secant stiffness* and *compliance* matrices are always *symmetric*.
5. *Thermodynamic laws* are *satisfied.* No energy is generated for load-unload loops.

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Any other proposals are welcome for discussing!

http://www.uni-weimar.de/cms/universitaet/fakultaet-bauingenieurwesen/professuren/studium/master-of-science/studiengang-bauingenieurwesen/constitutive-models.html