- ✓ Tractions, stress and equilibrium
- ✓ Stress tensor
- ✓ Tensor‘s invariants (2nd order)
  - ✓ Stress tensor invariants
- ➢ Deformation / strain tensor

http://www.uni-weimar.de/cms/Constitutive-models.5613.0.html?&L=%21

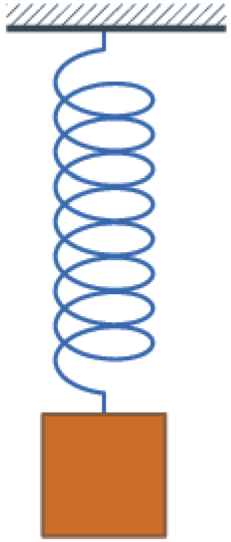
Next:

Position vector – displacement vector in Lagrangian and Eulerian description (very brief)

Deformation

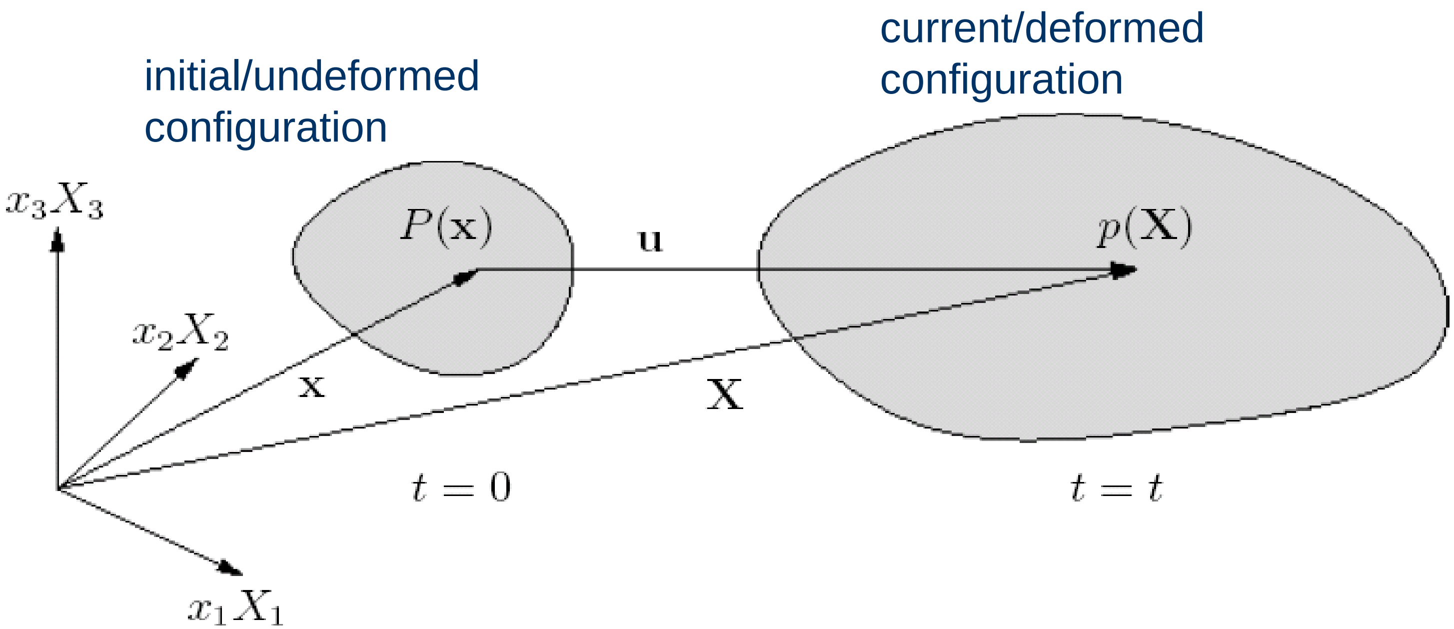
Strain tensor

Some geometrical representations



Constitutive model 1: Generalized Hooke's law

Position vector – displacement vector
Lagrangian and Eulerian description



Strain analysis concerns a geometrical problem since it studies the geometrical expression of the body deformation due to apllied forces.

$\Downarrow$

This analysis is not related to the material properties and it is NOT leading to constitutive relations.

Rigid body motion

If the distance between every pair of point (particles) in the body ramains constant during the body motion, then the body is said to experience rigid body motion

$$\Downarrow$$

The displacements of the rigid body consist of translations and rotations

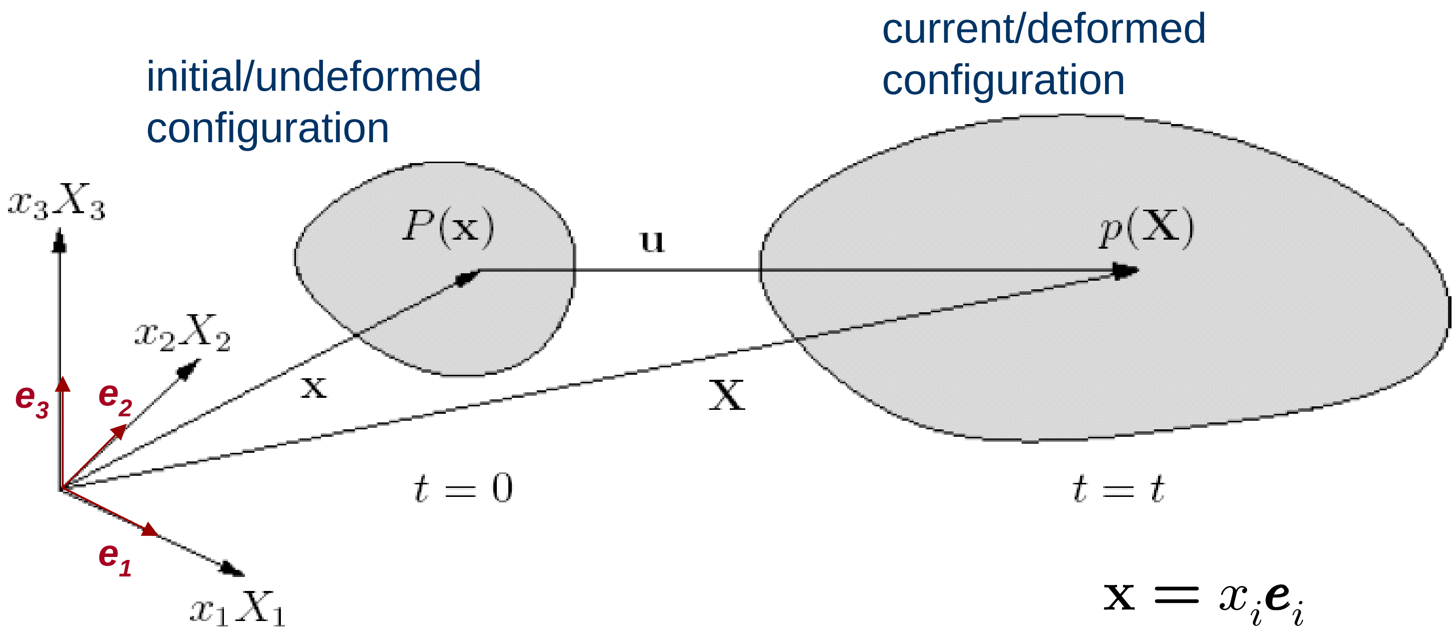
Deformed body motion

If the relative position of any two points (particles) is changed during the body motion then the body is said to be deformed.

$$\Downarrow$$

Displacements in a deformed body are given as:

$$\mathbf{u} = \mathbf{X} - \mathbf{x} \qquad u_i = X_i - x_i.$$



$$\mathbf{x} = x_i \boldsymbol{e}_i$$

$$\mathbf{X} = X_i \boldsymbol{e}_i$$

e_1, e_2, e_3 -> defined (fixed) normalized 3D orthogonal basis

The movement of the point (particle) *P* is called to be known (defined) if at each time *t* the relation between **X** and **x** is a known vector-function:

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) = \boldsymbol{\phi}(\mathbf{x}, t)$$

* these are 3 scalar relations

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) = \boldsymbol{\phi}(\mathbf{x}, t) \qquad \text{(L1)}$$

Lagrangian description; Lagrange (material) coordinates-curvilinear, nonorthogonal coordinates
(only at $t=t_0$ the Lagrangian coordinates are Cartesian)

The mapping (L1) is unique and invertible if the its Jacobian is such that:

$$J = \left| \frac{\partial X_i}{\partial x_j} \right| = |X_{i,j}| \neq 0$$

Inversion of (L1) gives:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \qquad \text{(E1)}$$

Eulerian description; Euler (spatial) coordinates – fixed Cartesian coordinate system

Displacement and velocity vectors (definition):

$$\mathbf{u} = \mathbf{X} - \mathbf{x} \qquad + \text{(L1)}$$

used by Lagrange method as basic kinematical characteristic

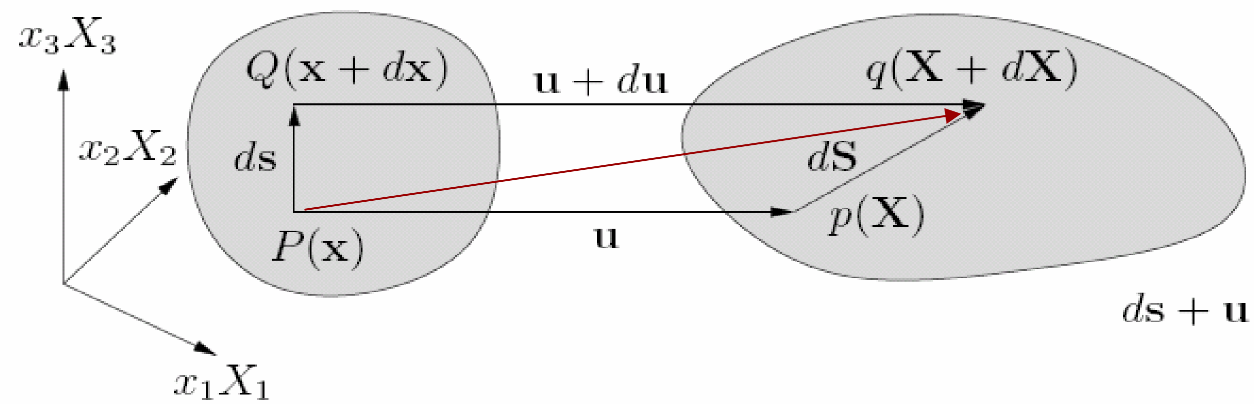
$$\mathbf{v} = \frac{\partial \mathbf{X}}{\partial t} \qquad + \text{(E1)}$$

used by Euler method as basic kinematical characteristic

Strain tensor

initial/undeformed configuration

current/deformed configuration



fibre PQ -> fibre pq

$$d\mathbf{s}+\mathbf{u}+d\mathbf{u}-d\mathbf{S}-\mathbf{u}=\mathbf{0}$$

$$\Rightarrow d\mathbf{u}=d\mathbf{S}-d\mathbf{s}.$$

$$|d\mathbf{s}|^2 = dx_i dx_i$$

$$|d\mathbf{S}|^2 = dX_i dX_i$$

expression of the distance → by the Jacobian of the mapping

$$|d\mathbf{s}|^2 = dx_i dx_i = \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} dX_j dX_k$$

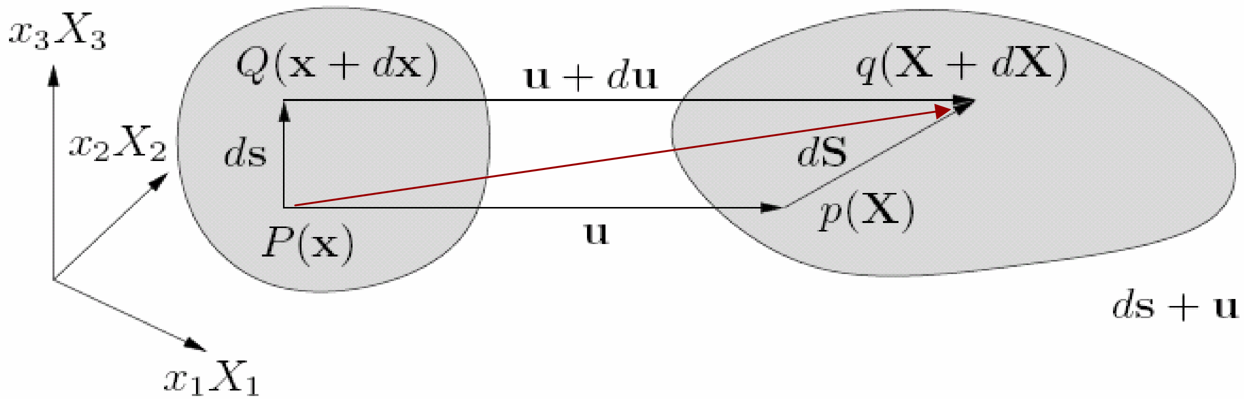
$$|d\mathbf{S}|^2 = dX_i dX_i = \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k} dx_j dx_k.$$

* deformation gradient $F_{ij} = X_{i,j}$

Strain tensor

initial/undeformed configuration

current/deformed configuration



fibre PQ -> fibre pq

$$d\mathbf{s} + \mathbf{u} + d\mathbf{u} - d\mathbf{S} - \mathbf{u} = \mathbf{0}$$

$$\Rightarrow d\mathbf{u} = d\mathbf{S} - d\mathbf{s}.$$

The distance in material (Lagrangian) coordinates:

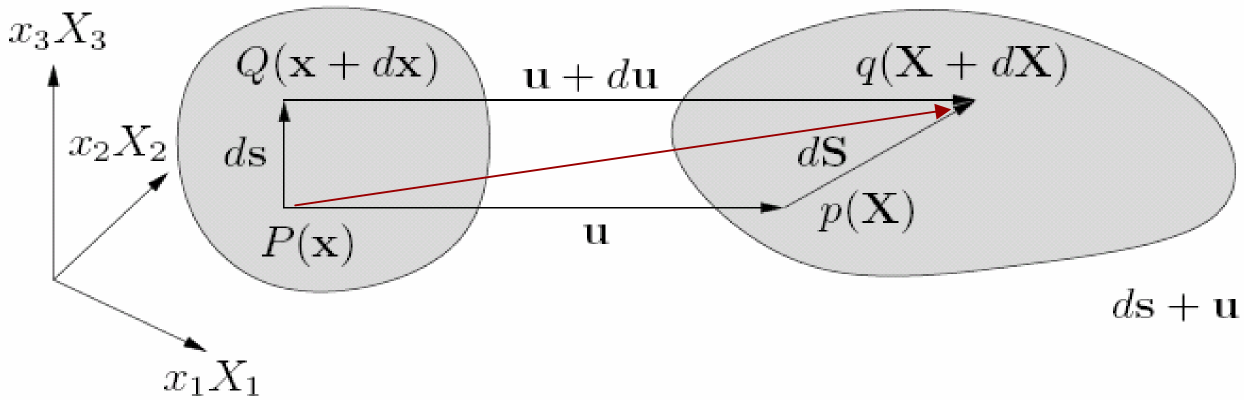
$$\begin{aligned} |d\mathbf{S}|^2 - |d\mathbf{s}|^2 &= X_{i,j} X_{i,k} dx_j dx_k - dx_i dx_i \\ &= \underbrace{\left(X_{i,j} X_{i,k} - \delta_{jk}\right)}_{=2\varepsilon^L_{jk}} dx_j dx_k \end{aligned} \qquad \text{(L41)}$$

Green or Lagrangian strain tensor

Strain tensor

initial/undeformed configuration

current/deformed configuration



$$d\mathbf{s} + \mathbf{u} + d\mathbf{u} - d\mathbf{S} - \mathbf{u} = \mathbf{0}$$

$$\Rightarrow d\mathbf{u} = d\mathbf{S} - d\mathbf{s}.$$

The distance in spatial (Eulerian) coordinates:

$$\begin{aligned} |d\mathbf{S}|^2 - |d\mathbf{s}|^2 &= dX_i dX_i - \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} dX_j dX_k \\ &= \underbrace{\left(\delta_{jk} - x_{i,j} x_{i,k}\right)}_{=2\varepsilon^E_{jk}} dX_j dX_k \qquad \text{(E42)} \end{aligned}$$

Euler or Almansi strain tensor

Strain tensor

represents the deformation of a point (particle) vicinity

Green or Lagrangian strain tensor

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial X_i}{\partial x_k} - \frac{\partial x_i}{\partial x_k} = X_{i,k} - \delta_{ik}$$

$$\Rightarrow X_{i,k} = u_{i,k} + \delta_{ik}$$

$\Rightarrow$ (L41)

$$\begin{aligned} \varepsilon^L_{jk} &= \frac{1}{2}[(u_{i,j} + \delta_{ij})(u_{i,k} + \delta_{ik}) - \delta_{jk}] \\ &= \frac{1}{2}[u_{i,j}u_{i,k} + u_{i,j}\delta_{ik} + \delta_{ij}u_{i,k} + \delta_{jk} - \delta_{jk}] \\ &= \frac{1}{2}[u_{k,j} + u_{j,k} + \underbrace{u_{i,j}u_{i,k}}] \end{aligned}$$

Euler or Almansi strain tensor

$$\frac{\partial u_i}{\partial X_k} = \frac{\partial X_i}{\partial X_k} - \frac{\partial x_i}{\partial X_k} = \delta_{ik} - x_{i,k}$$

$$\Rightarrow x_{i,k} = \delta_{ik} - u_{i,k}$$

$\Rightarrow$ (E42)

$$\begin{aligned} \varepsilon^E_{jk} &= \frac{1}{2}[\delta_{jk} - (\delta_{ij} - u_{i,j})(\delta_{ik} - u_{i,k})] \\ &= \frac{1}{2}[u_{k,j} + u_{j,k} - \underbrace{u_{i,j}u_{i,k}}] . \end{aligned}$$

For small displacement gradient (infinitesimal (small) deformations) Lagrangian and Euler strain tensors coincide (also the methods of description of motion):

$$\varepsilon_{ij} = \varepsilon^L_{ij} = \varepsilon^E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

* symmetric 2nd order tensor

Current summary:

6 unknowns σ_{ij} $\longrightarrow$ equilibrium equation gives 3 relations

6 unknowns ε_{ij} $\longrightarrow$ 6 kinematic equations relate strain tensor components to 3 displacement components

$$u_i\,,\quad i,j=1,2,3 \qquad\qquad \varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

Integrate the system of 6 equations to determine the 3 displacement components

$$\Downarrow$$

Compatibility equations (small strains)

$$\varepsilon_{ij,km}+\varepsilon_{km,ij}-\varepsilon_{ik,jm}-\varepsilon_{jm,ik}=0$$

1. $\dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\dfrac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\dfrac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$
2. $\dfrac{\partial^2\varepsilon_{22}}{\partial x_3^2}+\dfrac{\partial^2\varepsilon_{33}}{\partial x_2^2}=2\dfrac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}$
3. $\dfrac{\partial^2\varepsilon_{33}}{\partial x_1^2}+\dfrac{\partial^2\varepsilon_{11}}{\partial x_3^2}=2\dfrac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_1}$
4. $\dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial\varepsilon_{23}}{\partial x_1}+\dfrac{\partial\varepsilon_{31}}{\partial x_2}+\dfrac{\partial\varepsilon_{12}}{\partial x_3}\right)=\dfrac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3}$
5. $\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial\varepsilon_{23}}{\partial x_1}-\dfrac{\partial\varepsilon_{31}}{\partial x_2}+\dfrac{\partial\varepsilon_{12}}{\partial x_3}\right)=\dfrac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}$
6. $\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial\varepsilon_{23}}{\partial x_1}+\dfrac{\partial\varepsilon_{31}}{\partial x_2}-\dfrac{\partial\varepsilon_{12}}{\partial x_3}\right)=\dfrac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}$