## Pilot test (name, e-mail, main topic of interst):

1. $$f(x_1, x_2, x_3) = 3x_1 + x_1 e^{x_2} + x_1 x_2 e^{x_3}$$

   (a) $\textbf{grad}\, f = ?$ (b) $\textbf{grad}\, f(3,1,0) = ?$

2. Is the equation of motion a constitutive equation?
3. Give the name of at least one constitutive law you know.
3. For the tensor $T_{ij}$ it holds $T_{ij} = -T_{ji} \quad i, j = 1,2,3$ .
   Write the tensor $T_{ij}$ as a matrix.
   Give the rank (order) of the tensor $T_{ij}$ .
4. In 2D a coordinate transform $x_i \longrightarrow x'_i$ is defined with the angle between $x_1$ and $x'_1$ equal to $\pi/2$ . Give the rotational matrix $\alpha_{ij}$ .

**Engineering Problem**

**Boundary and initial value problem**
**geometry, applied constrains, forces etc.**

**Starting point for deriving partial differential equations (PDE) lies in the conservative laws of physics**

**Conservation of mass (Continuity equiation)**

general laws

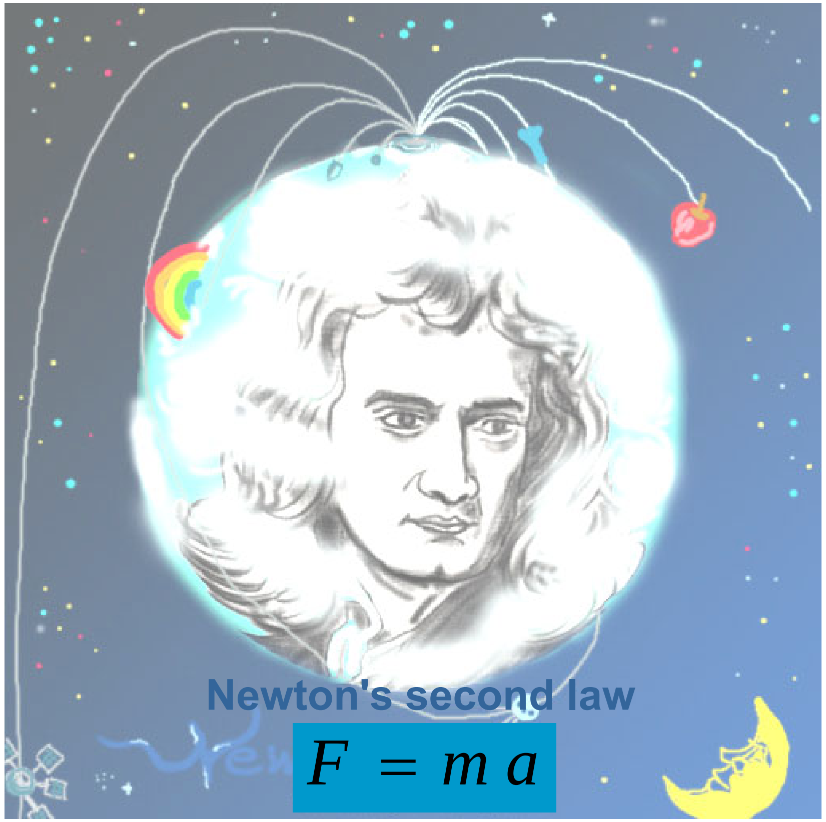
**Conservation of energy (Equation of equilibrium/ motion)**

**Add constitutive relationships (stress-strain relationships)**

**SOLVE the PDE – BVP**
numerically: FE; FD; FV etc

nonlinear PDE,
geometrical nonlinearity:
analytical solution rarely possible

**PREDICTIONS**




**Mass** is a fundamental **concept** in physics,
roughly corresponding to the intuitive idea of "how much matter there is in an object"

In physics, **acceleration** is defined as the rate of change of velocity,

**Velocity** is defined as the rate of change of position.

**Position** - location in a coordinate system

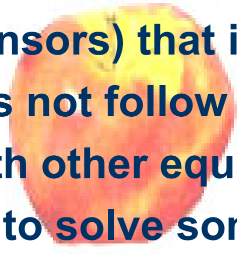
Equation of motion and does it present a constitutive law?

NO – it is a general law and it is material independent!

Everything „moves“ or is in a „static“ equilibrium.

Constitutive <=> material law

**a constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material or substance, and does not follow directly from physical law. It is combined with other equations that do represent physical laws to solve some physical problem**



Voting recapitulation on 15th of October 2007: YES 3; **NO 4**; X 3

Voting recapitulation on 19th of December 2007: coming soon

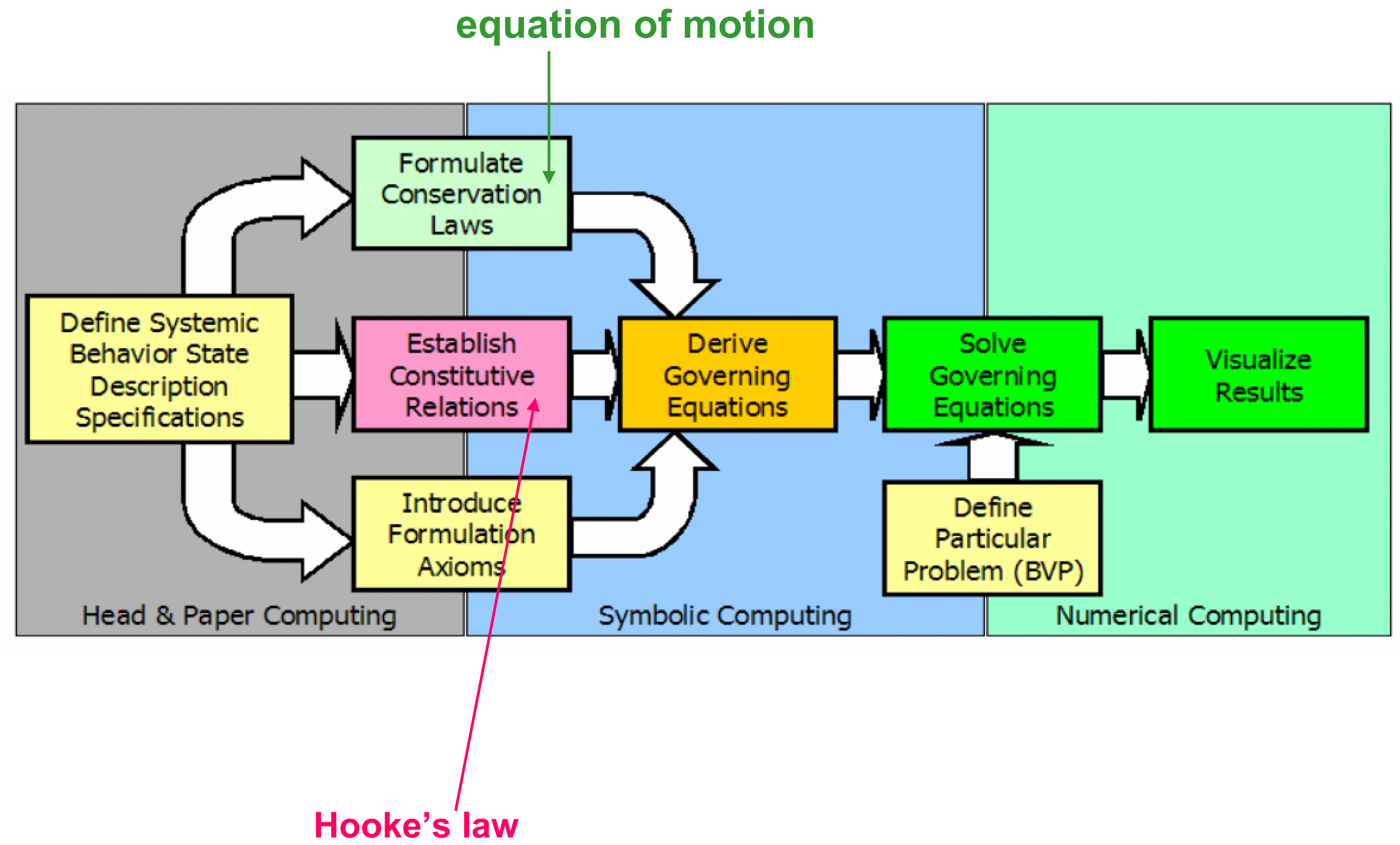
## i.e., conceptual and mathematical mechanics
## Truesdell, Noll, etc

The modern mechanical theories regard **bodies** as subject to general laws applying to all types of materials, laws that characterize geometry of space, time and motion, the structure of material bodies, the nature of systems of forces and the relation of force to motion.

Mechanics uses the term **constitutive assumption** to refer to the special laws for particular materials, since these laws reflect assumptions about the constitution of the material.

Mechanical practice depends critically on these special laws.

**Constitutive assumptions – relations – laws – equations are the TOPIC of this lecture course**




Michopoulos, Farhat, Fish, 2005

Tractions, stress and equilibrium

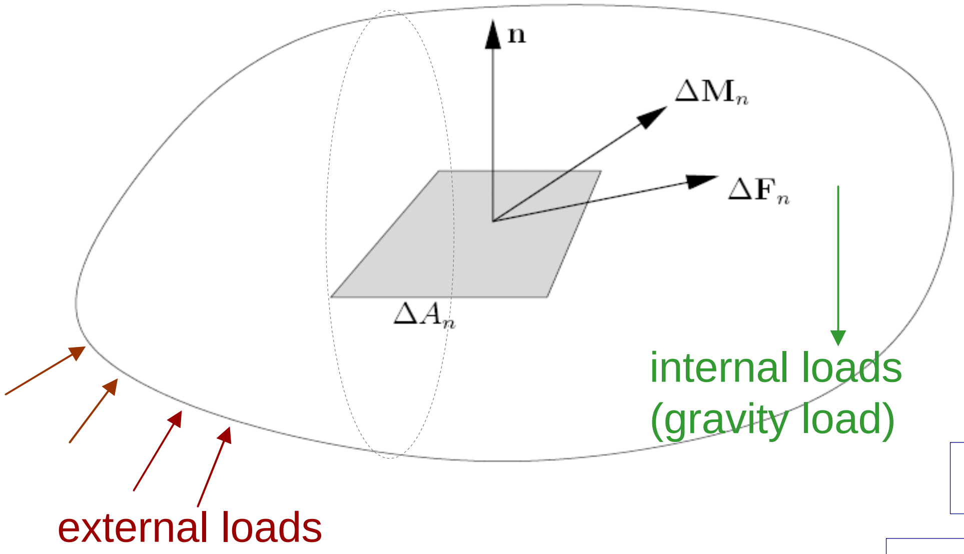
Stress tensor

Deformation / strain tensor

Tensor‘s invariants (2nd order)

# Traction and couple-stress vector

deformable body under load (external + internal)




$$\lim_{\Delta A_n \to 0} \frac{\Delta \mathbf{F}_n}{\Delta A_n} = \frac{d\mathbf{F}_n}{dA_n} = \mathbf{t}_n$$

traction vector
limiting force intensity

$$\lim_{\Delta A_n \to 0} \frac{\Delta \mathbf{M}_n}{\Delta A_n} = \frac{d\mathbf{M}_n}{dA_n} = \mathbf{C}_n$$

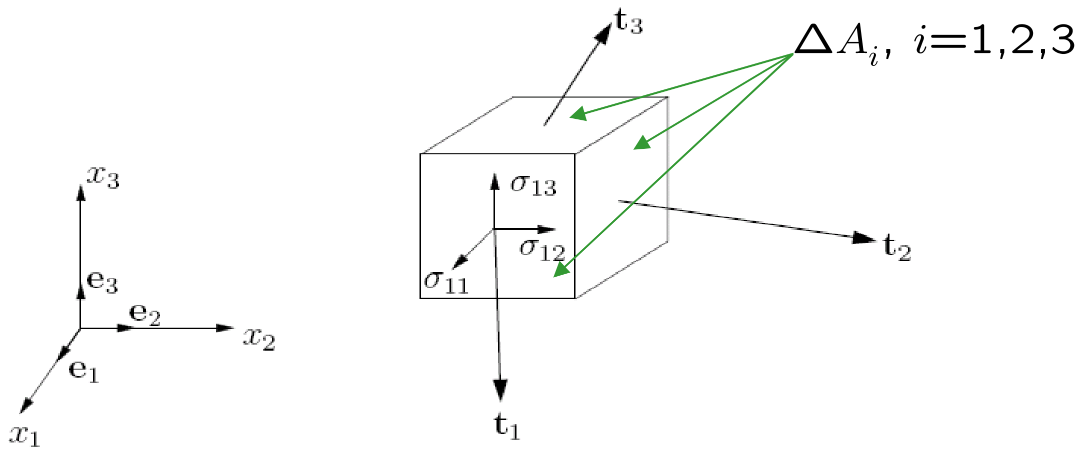
couple-stress vector

=0 if no particle rotation is considered

else the continua are called Cosserat-Continua and it is not included in this lecture course

$\Delta A_n$ is small and => $\Delta F_n$ is almost constant over $\Delta A_n$

## Stress tensor

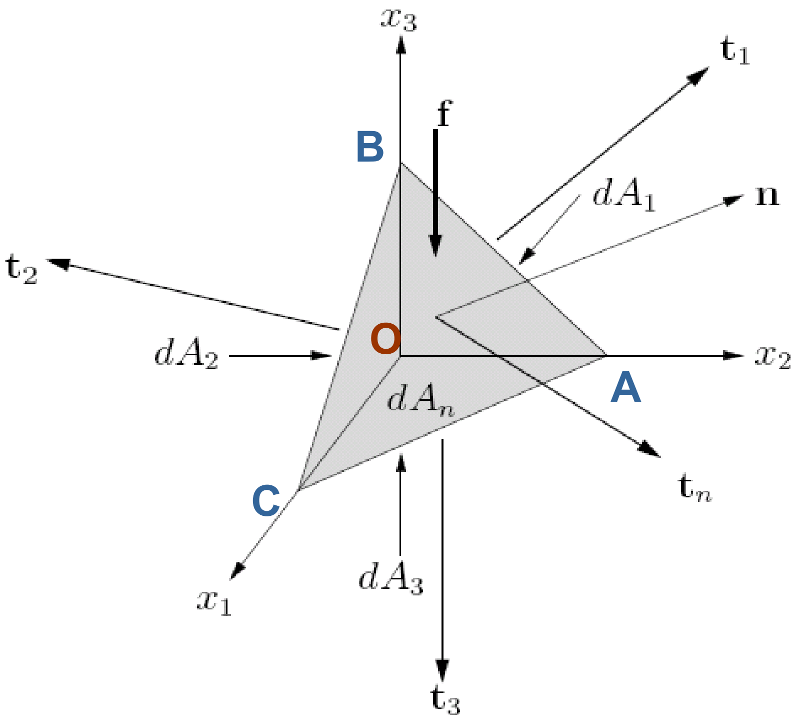



$$\mathbf{t}_i = \sigma_{i1}\mathbf{e}_1 + \sigma_{i2}\mathbf{e}_2 + \sigma_{i3}\mathbf{e}_3$$

$$\mathbf{t}_i = \sigma_{ij}\mathbf{e}_j .$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \sigma_{ij} ,$$

## Stress tensor




$$\mathrm{d}x_i\,\boldsymbol{e}_i \cdot \boldsymbol{n} = h$$

**no summation over i**

$$\frac{1}{3}\mathrm{d}A_\beta\,\mathrm{d}x_\beta = \frac{1}{3}\mathrm{d}A_n\,h, \quad \beta = 1,2,3$$

$\Downarrow$ **no summation over β**

$$\mathrm{d}A_\beta = \mathrm{d}A_n\,\boldsymbol{e}_\beta \cdot \boldsymbol{n} = \mathrm{d}A_n\,n_\beta$$

Equilibrium of forces at the tetrahedron:

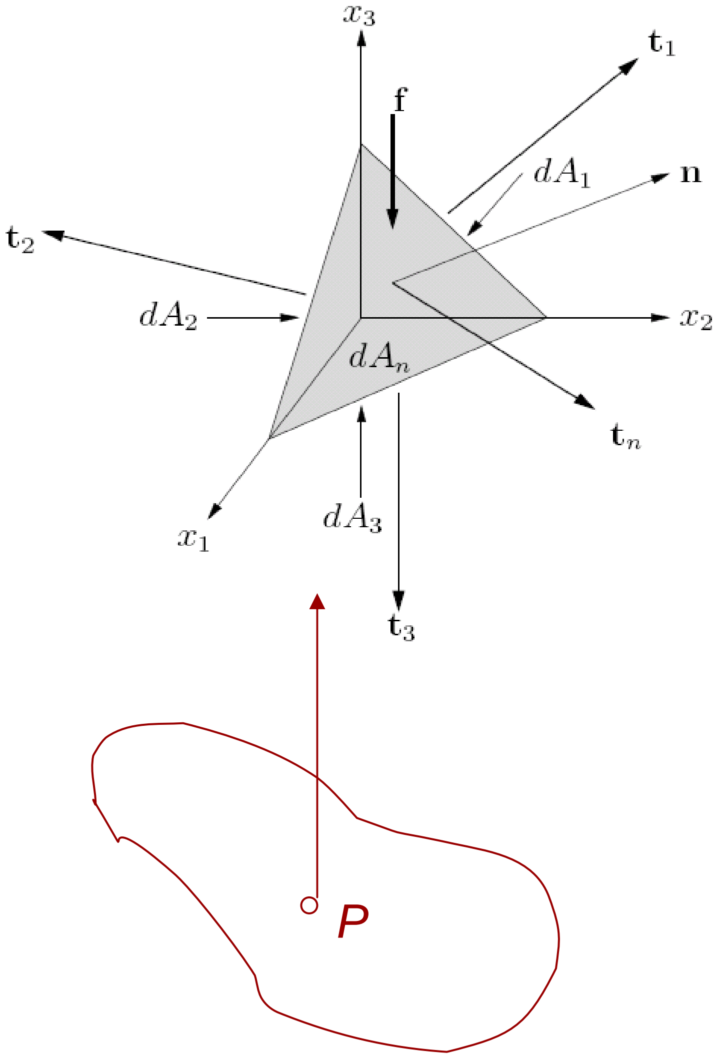
$$\boldsymbol{t}_n \mathrm{d}A_n - \boldsymbol{t}_i \mathrm{d}A_i + \boldsymbol{f}\tfrac{1}{3}\,h\,\mathrm{d}A_n = 0 \;\Rightarrow$$

by definition

$$\boldsymbol{t}_n - \boldsymbol{t}_i n_i + \boldsymbol{f}\tfrac{1}{3}\,h = 0,\; h \to 0 \;\Rightarrow \boldsymbol{t}_n = \boxed{\boldsymbol{t}_i} n_i = \boxed{\sigma_{ij}\boldsymbol{e}_j} n_i$$

$$\boxed{t_i = \sigma_{ij} n_j} \Leftarrow \boldsymbol{t}_n = t_i \boldsymbol{e}_i$$

**summation over i**




$$\boldsymbol{t}_n = t_i \boldsymbol{e}_i$$

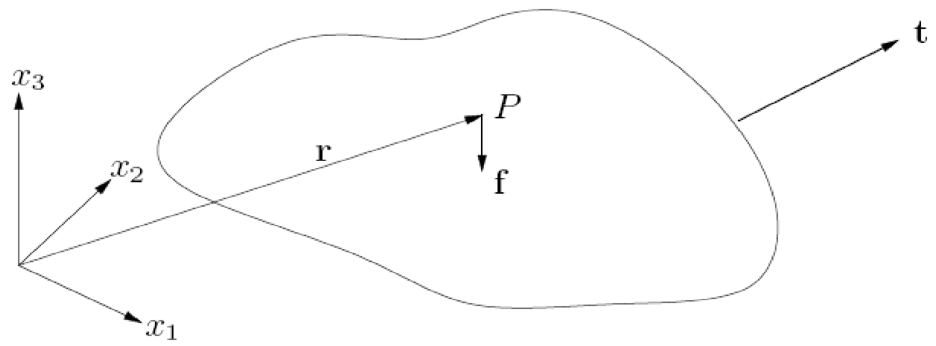
$$t_i = \sigma_{ij} n_j$$

$$\Downarrow$$

At a given body point *P* the tranction in a given direction $\boldsymbol{n}$ is uniquely determined by the stress tensor $\sigma_{ij}$

Stress tensor is a 2nd order tensor => it is represented by 9 nimbers:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \sigma_{ij}$$




Equilibrium is a general property concerned with the conservation of energy => laws related to it are general laws, material independent and NOT constitutive laws.

1. Linear momentum principle: consider arbitrary body with volume *V* and surface boundary *A* =>

$$\int_V \mathbf{f}\, dV + \int_A \mathbf{t}\, dA = \int_V \rho \frac{d^2}{dt^2}\mathbf{u}\, dV$$

with displacement vector $\mathbf{u}$ and density $\rho$.

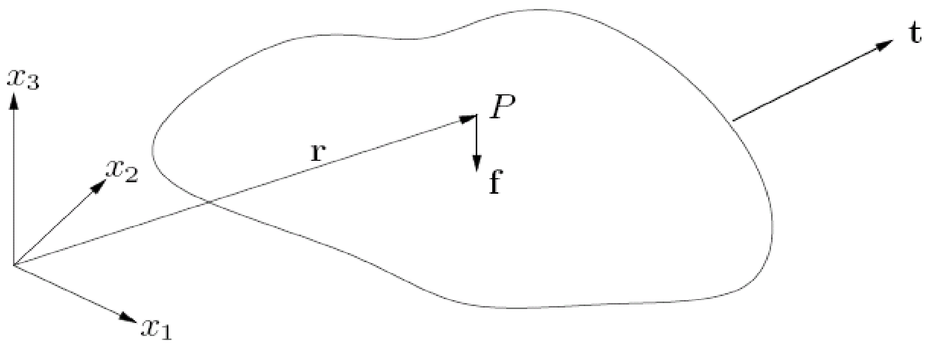
$$\Downarrow$$

Equation of motion:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i$$

As *i=1,2,3* => these are 3 equations for the unknown stress




Equilibrium is a general property concerned with the conservation of energy => laws related to it are general laws, material independent and NOT constitutive laws.

2. Momentum equilibrium principle: consider arbitrary body with volume *V* and surface boundary *A* =>

$$\int_V (\mathbf{r} \times \mathbf{f})\, dV + \int_A (\mathbf{r} \times \mathbf{t})\, dA = \int_V (\mathbf{r} \times \rho \ddot{\mathbf{u}})\, dV$$

with displacement vector $\mathbf{u}$ and density $\rho$.

$$\Downarrow$$

Gives that stress tensor is a symmetric 2nd order tensor:

$$\sigma_{ij} = \sigma_{ji} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad \Rightarrow$$

6 numbers define the stress tensor at a given body point.

1. Linear momentum principle

$$\nabla \cdot \boldsymbol{\sigma} + \mathrm{f} = \rho \ddot{\mathrm{u}}$$

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i$$

As *i=1,2,3* =>
these are 3 equations for the unknown stress tensorcomponents

2. Momentum equilibrium principle

$$\sigma_{ij} = \sigma_{ji} \qquad \overset{!}{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T} \quad \Rightarrow$$

6 numbers define the stress tensor at a given body point.

$$\Downarrow$$

There are 3 equations missing to complete the system of PDE for the unknown stress tensor. These missing equations will come due to the **constitutive relations**.

## Principal axes, principal stresses

Question: Is there in any body at any particular body point a plane where the area element experiences only normal stresses?

Answer: It is known from the linear algebra that there exists an orthogonal transformation which transforms a symmetric matrix to a diagonal one => yes and for determining these planes the eigenvalue problem has to be solved, namely to solve:

**principal values**

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| = 0 \quad \Longleftrightarrow \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{pmatrix}$$

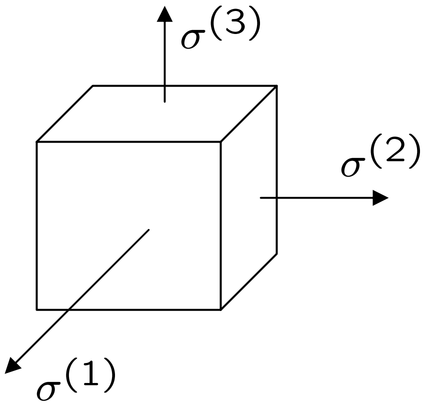
$\sigma^{(k)}$ - (eigenvalues) principal stresses

↑ In principal axes

**principal axes**

$$\left. \begin{array}{r} (\sigma_{ij} - \sigma^{(k)}\delta_{ij})n_j \overset{!}{=} 0 \\ n_i^{(k)} n_i^{(k)} = 1 \end{array} \right\} \quad \Longleftrightarrow$$

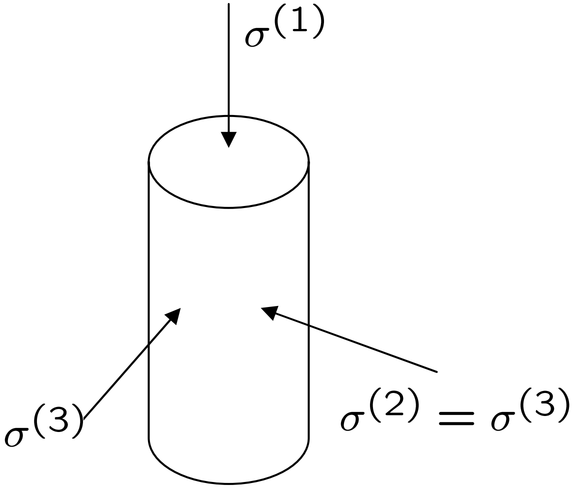
to determine (eigenvectors) principal directions

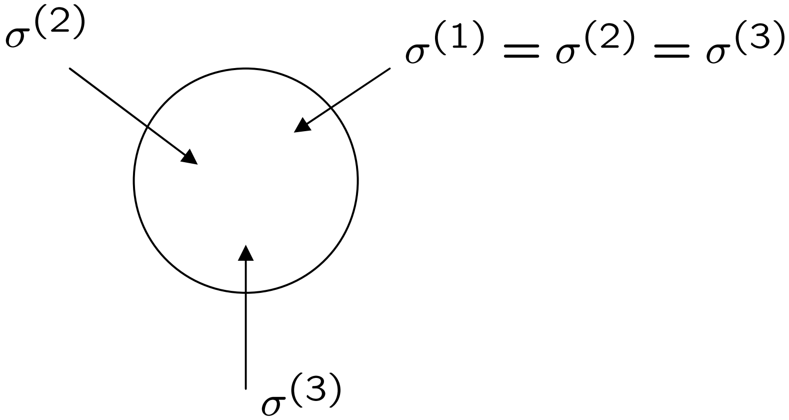



## Principal axes, principal stresses

1. All principal directions are orthogonal if the three principal stresses are distinct => there are 3 numbers (principal stresses) and 3 angles (directions) for determining the stress in the principal (main) coordinate system.

$$\sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)}$$




! $\sigma^{(1)} > \sigma^{(2)} = \sigma^{(3)}$

## Principal axes, principal stresses

2. All the three principal stresses and the three principal directions are REAL. (this can be easily proved using the symmetry property of the stress tensor and the fact that the eigenvalue equation is a polynomial of 3rd order).

Summary: If the three roots of the eigenvalue problem are different => there exist three mutually orthogonal directions at point *P* such that area elements perpendicular to these directions experience only NORMAL stresses and these directions are called principal directions at point *P*. The corresponding normal stresses are called principal stresses.

## Stress invariants, deviatoric and spherical decomposition

! In general, the stress tensor at a distinct point differs in its form for different coordinate systems.

Previous knowledge: Eigenvalues are invariant regarding orthogonal coordinate transformations (rotations)

$$\Downarrow$$

1. Principal stresses are invariants of the stress tensor.

2. As the eigenvalue equation is form-invariant regarding orthogonal transformation of the coordinate system => its coefficients are also being stress invariants:

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| = (\sigma^{(k)})^3 - I_1(\sigma^{(k)})^2 + I_2\sigma^{(k)} - I_3 \stackrel{!}{=} 0$$

$$I_1 = \sigma_{ii} = tr\boldsymbol{\sigma} \qquad I_1 = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}$$

$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \qquad I_2 = (\sigma^{(1)}\sigma^{(2)} + \sigma^{(2)}\sigma^{(3)} + \sigma^{(3)}\sigma^{(1)})$$

$$I_3 = |\sigma_{ij}| = \det\boldsymbol{\sigma} \qquad I_3 = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)},$$

(coefficients depend on the stress tensor components)

$$I_1 = \sigma_{ii} = tr\boldsymbol{\sigma}$$
$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})$$
$$I_3 = |\sigma_{ij}| = \det\boldsymbol{\sigma}$$

$I_1$, $I_2$, and $I_3$ are **stress invariants** as they do not change in value when the axes are rotated to new positions.

(Previous knowledge: **Invariants** are scalar functions of tensors that by definition have the same value no matter the coordinate system to which they are referenced. )

!

It is important to understand **stress invariants** since they play important role in developing constitutive equations.

!

## Deviatoric and spherical decomposition

$$\boldsymbol{\sigma} = \boldsymbol{s} + \sigma_m \boldsymbol{I} = \boldsymbol{s} + \tfrac{1}{3} I_1 \boldsymbol{I}$$

$$\sigma_{ij} = s_{ij} + \sigma_m \delta_{ij} = s_{ij} + \tfrac{1}{3} I_1 \delta_{ij} = s_{ij} + \tfrac{1}{3} \sigma_{kk} \delta_{ij}$$

spherical part of the stress tensor $\boldsymbol{\sigma}$

$$\mathbf{s} = \begin{pmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_m \end{pmatrix}$$

deviatoric part of the stress tensor $\boldsymbol{\sigma}$

Other in common use forms for the three stress invariants:

$$\begin{cases} I_1 = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases}$$

- $I_1$ → first stress invariant (polynomial of degree 1)
- $I_2$ → second stress invariant (polynomial of degree 2)
- $I_3$ → third stress invariant (polynomial of degree 3)

$$\begin{cases} I_\sigma = \operatorname{tr}\boldsymbol{\sigma} = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 = I_1 & \text{(1st)} \\ II_\sigma = \operatorname{tr}\boldsymbol{\sigma}^2 = \sigma_{ij}\sigma_{ij} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = I_1^2 - 2\,I_2 & \text{(2nd)} \\ III_\sigma = \operatorname{tr}\boldsymbol{\sigma}^3 = \sigma_{ik}\sigma_{kj}\sigma_{ji} = \sigma_1^3 + \sigma_2^3 + \sigma_3^3 = I_1^3 - 3\,I_1\,I_2 + 3I_3 & \text{(3rd)} \end{cases}$$

stress deviator invariants
(1st is equal to 0)

$$\begin{cases} II_s = \frac{1}{2} s_{ij}s_{ij} = \frac{1}{2}\left(II_\sigma - \frac{1}{3}I_\sigma^2\right) & \text{(2nd)} \\ III_s = \frac{1}{3} s_{ij}s_{jk}s_{ki} = \frac{1}{3}\left(III_\sigma - I_\sigma II_\sigma - \frac{2}{9}I_\sigma^3\right) & \text{(3rd)} \end{cases}$$

Elementary: Any combination of any stress invariants is a stress invariant.

Not so elementary: the polynomial representation for the stress invariants is of max 3rd degree.
Any stress invariant can be represented using $I_1$, $I_2$, $I_3$.

* Notation for the principal stresses that is used more often is with subscript index:

$$\sigma^{(i)} \leftrightarrow \sigma_i$$

Next:

Position vector – displacement vector in Lagrangian and Eulerian description (very brief)

Deformation

Strain tensor

Some geometrical representations

**Constitutive model 1**: Generalized Hooke‘s law