Constitutive models: Incremental plasticity – Drücker's postulate

$$\boxed{d\boldsymbol{\varepsilon}^p \sim \text{grad}\, f} \qquad \Rightarrow \qquad D_{ij} \sim \text{grad}\, f$$

$$\Downarrow$$

$$D_{ij} = k \frac{\partial f}{\partial s_{ij}}$$

$$\Downarrow$$

$$de^p_{ij} = k \frac{\partial f}{\partial s_{ij}} \frac{\partial f}{\partial s_{kl}} ds_{kl}$$

if

$$\frac{\partial f}{\partial s_{mn}} ds_{mn} > 0$$

$$\Downarrow$$

consistency condition

$$D_{ij} \frac{\partial f}{\partial e^p_{ij}} = -1 - \frac{\dfrac{\partial f}{\partial \chi^p_k} d\chi^p_k}{\dfrac{\partial f}{\partial s_{ij}} ds_{ij}} \xrightarrow{\dfrac{\partial f}{\partial e^p_{mn}} \neq 0} k = -\frac{\dfrac{\partial f}{\partial s_{kl}} ds_{kl} + \dfrac{\partial f}{\partial \chi^p_k} d\chi^p_k}{\dfrac{\partial f}{\partial e^p_{ij}} \dfrac{\partial f}{\partial s_{ij}} \dfrac{\partial f}{\partial s_{kl}} ds_{kl}}$$

associated plastic law, associated plasticity -
plastic flow law associated with the limit (loading) surface
Prager's assumptions + Drucker's postulate

Constitutive models: Incremental plasticity – Drücker’s postulate

$$de_{ij}^{p} = \boxed{k \frac{\partial f}{\partial s_{kl}} ds_{kl}} \frac{\partial f}{\partial s_{ij}} \qquad \text{if} \qquad \frac{\partial f}{\partial s_{mn}} ds_{mn} > 0$$

$$\Downarrow$$

Drucker‘s postulate gives as a result

associated plasticity theory

associated plastic law

plastic flow law associated with the limit/loading surface

Constitutive models: Incremental plasticity – Flow rule

Loading process $\Rightarrow$ Plastic deformation

Describe the

stress-strain relationship

$\updownarrow$

Determine:

1. direction
2. magnitude

$\left.\right\}$ of plastic strain increment

de_{ij}^{p} $d\varepsilon_{ij}^{p}$

+

flow rule concept

$\hookrightarrow$ defines the direction of $d\varepsilon_{ij}^{p}$

+

consistency condition

$\hookrightarrow$ defines the magnitude of $d\varepsilon_{ij}^{p}$

Constitutive models: Incremental plasticity – Flow rule – Plastic potential

Plastic potential function $g\left(\sigma_{ij},\, \alpha_{ij},\, \Delta T\right)$

internal variables, e.g. $d\varepsilon_{ij}^{p}$

By analogy with ideal fluid-flow models:

normality condition

$$d\varepsilon_{ij}^{p} = d\lambda \boxed{\frac{\partial g}{\partial \sigma_{ij}}}$$

direction of the plastic strain increment (not normalized!)

$d\lambda$ → non-negative scalar function that is varying throughout the plastic deformation history (time like parameter)

Plastic potential surface: $g\left(\sigma_{ij},\, \alpha_{ij},\, \Delta T\right) = 0$

Constitutive models: Incremental plasticity – Flow rule – Plastic potential

The flow rule is

associated

$$g = f$$

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

normal to the loading surface
following from the Drucker's postulate

non-associated

$$g \neq f$$

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}$$

normal to the potential surface

$$de^p_{ij} = \boxed{k \frac{\partial f}{\partial s_{kl}} ds_{kl}} \frac{\partial f}{\partial s_{ij}}$$

! in the non-associated plasticity theories the Drücker's postulate can be violated

Constitutive models: Incremental plasticity – Flow rule – Perfectly plastic

Incremental stress-strain relationship for perfect plastic material

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij}$$

$$d\sigma_{ij} = C_{ijkl} d\varepsilon^{e}_{kl} = \underset{\uparrow}{C_{ijkl}} \left(d\varepsilon_{kl} - \boxed{d\lambda} \frac{\partial g}{\partial \sigma_{kl}} \right)$$

tangent stiffness matrix (elastic)

Perfect plastic $\Rightarrow$ no hardening and on the loading surface are both the loading and the neutral states

loading criteria for perfect plastic materials:

$\hookrightarrow$

$f = 0$ and $\dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$ Plastic loading or neutral loading

$f = 0$ and $\dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0$ unloading

Constitutive models: Incremental plasticity – Flow rule – Perfectly plastic

Incremental stress-strain relationship for perfect plastic material

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

$$d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}^{e} = \underset{\uparrow}{C_{ijkl}} \left(d\varepsilon_{kl} - d\lambda \frac{\partial g}{\partial \sigma_{kl}} \right)$$

tangent stiffness matrix (**elastic**)

Perfect plastic $\Rightarrow$ no hardening and on the loading surface are both the loading and the neutral states

during the plastic deformation the stress point stays on the yield surface

→ consistency condition:

$$f\left(\sigma_{ij}\right) = 0 \quad \Rightarrow \quad f\left(\sigma_{ij} + d\sigma_{ij}\right) = f\left(\sigma_{ij}\right) + df\left(\sigma_{ij}\right) = 0$$

$$\Downarrow$$

$$\boxed{df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0}$$

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Constitutive models: Incremental plasticity – Flow rule – Perfectly plastic

Incremental stress-strain relationship for perfect plastic material

Perfect plastic:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$$

$$\updownarrow$$

$+$

$$\underline{d\sigma_{ij}} = C_{ijkl}\left(\boxed{d\varepsilon_{kl}} - d\lambda \frac{\partial g}{\partial \sigma_{kl}}\right) \Bigg| \cdot \frac{\partial f}{\partial \sigma_{ij}}$$

$$\boxed{d\lambda} = \frac{1}{H} \frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \boxed{d\varepsilon_{kl}} \qquad \text{(EqP)}$$

calculate

known/given

$$d\lambda = 0$$ if unloading or the right side in (EqP) = 0

$$H = \frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial g}{\partial \sigma_{kl}}$$

Constitutive models: Incremental plasticity – Flow rule – Perfectly plastic

Incremental stress-strain relationship for perfect plastic material

Perfect plastic:

$$d\sigma_{ij} = C_{ijkl}\left(\underline{d\varepsilon_{kl}} - d\lambda\,\frac{\partial g}{\partial \sigma_{kl}}\right)$$

$$d\lambda = \frac{1}{H}\,\frac{\partial f}{\partial \sigma_{ij}}\,C_{ijkl}\,\underline{d\varepsilon_{kl}}$$

$$\Downarrow$$

$$d\sigma_{ij} = C^{ep}_{ijkl}\,d\varepsilon_{kl}$$

function of the stress tensor

$$\boxed{C^{ep}_{ijkl} = C_{ijkl} - \frac{1}{H}\,H^{*}_{ij}\,H_{kl}}$$

Elastoplastic tangent stiffness matrix
(requested in e.g. FEM codes at the
Gaussian integration point).
Do not confuse with the algorithmic stiffness

$$H^{*}_{ij} = C_{ijmn}\,\frac{\partial g}{\partial \sigma_{mn}} \qquad H_{kl} = C_{klpq}\,\frac{\partial f}{\partial \sigma_{pq}}$$

Constitutive models: Incremental plasticity – Flow rule – Plastic potential

Von Mises type of plastic potential function

$$g\left(\sigma_{ij}\right) = \sqrt{J_2}$$

and plastic potential surface:

$$g\left(\sigma_{ij}\right) = \sqrt{J_2} - k = 0$$

$$\Downarrow$$

flow rule:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{s_{ij}}{2\sqrt{J_2}} \quad \Rightarrow \quad de_{ij}^{p} = d\lambda \frac{s_{ij}}{2\sqrt{J_2}}$$

$$d\varepsilon_{kk}^{p} = d\lambda \frac{s_{kk}}{2\sqrt{J_2}} = 0$$

The principal axes of plastic strain increment coincide with the principal axes of stress

Remarks:

The principal axes of the stress deviator coincide with the principal axes of the total stress as the spherical part of the (stress) tensor does not influence the principal directions of the tensor.

St Venant in 1870 stated that for rigid plastic case the principal axes of the strain increment coincide with the principal axes of stress

Constitutive models: Incremental plasticity – Flow rule – Plastic potential

Drücker-Prager type of plastic potential function

$$g\left(\sigma_{ij}\right) = \alpha I_1 + \sqrt{J_2}$$

and plastic potential surface:

$$g\left(\sigma_{ij}\right) = \alpha I_1 + \sqrt{J_2} = \text{const}$$

$$\boxed{g\left(I_1,\, J_2\right)}$$

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial g}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial g}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}}$$

flow rule:

$$de_{ij}^{p} = d\lambda \boxed{\frac{\partial g}{\partial J_2}} s_{ij} \qquad de_{ij}^{p} = d\lambda \boxed{\frac{1}{2\sqrt{J_2}}} s_{ij}$$

$$d\varepsilon_{kk}^{p} = 3\, d\lambda \boxed{\frac{\partial g}{\partial I_1}} \qquad d\varepsilon_{kk}^{p} = 3 \boxed{\alpha}\, d\lambda$$

Plastic potential depends on the first stress invariant -> volumetric plastic deformation different than 0

If   α=0   the von Mises plastic flow law will be recovered

Constitutive models: Incremental plasticity – Flow rule – Plastic potential

Isotropic linear elastic – perfectly plastic material

Drücker-Prager type of plastic potential function

$$g\left(\sigma_{ij}\right) = \alpha\, I_1 + \sqrt{J_2}$$

+ associated flow law supposes a loading function:

$$f\left(\sigma_{ij}\right) = \alpha\, I_1 + \sqrt{J_2} - k$$

$$d\sigma_{ij} = C^{ep}_{ijkl}\, d\varepsilon_{kl}$$

where

$$C^{ep}_{ijkl} = C_{ijkl} - \frac{1}{H} H^{*}_{ij}\, H_{kl}$$

with

$$H^{*}_{ij} = C_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \qquad H_{kl} = C_{klpq} \frac{\partial f}{\partial \sigma_{pq}} \qquad H = \frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial g}{\partial \sigma_{kl}}$$

For isotropic linear elastic case the constitutive fourth order stiffness tensor reads:

$$C_{ijkl} = \left(K - \frac{2}{3} G\right) \delta_{ij}\, \delta_{kl} + G \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

Constitutive models: Incremental plasticity – Flow rule

Isotropic linear elastic – perfectly plastic material

Drücker-Prager type of plastic potential function

$$g\left(\sigma_{ij}\right) = \alpha\, I_1 + \sqrt{J_2}$$

+ loading function: $g \equiv f$

$$f\left(\sigma_{ij}\right) = \alpha\, I_1 + \sqrt{J_2} - k$$

$$H^*_{ij} = C_{ijmn}\,\frac{\partial g}{\partial \sigma_{mn}} = H_{ij} = \frac{\partial f}{\partial \sigma_{nm}}\,C_{mnij}$$

$$H = \frac{\partial f}{\partial \sigma_{ij}}\,C_{ijkl}\,\frac{\partial g}{\partial \sigma_{kl}}$$

$$H_{ij} = 3\,\alpha\,K\,\delta_{ij} + \frac{G}{\sqrt{J_2}}\,s_{ij}$$

$$H = 9\alpha^2\,K + G$$

$$C^{ep}_{ijkl} = C_{ijkl} - \frac{1}{H}\,H^*_{ij}\,H_{kl}$$

Constitutive models: Incremental plasticity – Flow rule

$$H_{ij} = 3\,\alpha\,K\,\delta_{ij} + \frac{G}{\sqrt{J_2}}\,s_{ij} \qquad\qquad H = 9\alpha^2\,K + G$$

$$C_{ijkl} = \left(K - \frac{2}{3}G\right)\,\delta_{ij}\,\delta_{kl} + G\,\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\Downarrow$$

$$C^{ep}_{ijkl} = C_{ijkl} - \frac{1}{H}\,H_{ij}\,H_{kl}$$

Constitutive models: Incremental plasticity – Flow rule

Drücker-Prager type of plastic potential function $g\left(\sigma_{ij}\right) = \alpha I_1 + \sqrt{J_2}$

+ associated flow law gives loading function: $f\left(\sigma_{ij}\right) = \alpha I_1 + \sqrt{J_2} - k$

$$d\lambda = \frac{1}{H} \frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \, d\varepsilon_{kl} \qquad d\lambda = \frac{1}{H} H_{kl} \, d\varepsilon_{kl}$$

$$d\lambda = \frac{1}{9\alpha^2 K + G} \left(3\alpha K d\varepsilon_{kk} + \frac{G}{\sqrt{J_2}} s_{kl} de_{kl} \right)$$

Constitutive models: Incremental plasticity – Flow rule – Hardening plasticity

loading function: $f\left(\sigma_{ij}, \boxed{\varepsilon^p_{ij}}\right)$

$$\Downarrow$$

$$f\left(\sigma_{ij},\, \varepsilon^p_{ij}\right) = 0$$

$$f\left(\sigma_{ij} + d\sigma_{ij},\, \underline{\varepsilon^p_{ij} + d\varepsilon^p_{ij}}\right) = 0$$

the new position of the loading surface

$$\Downarrow$$

term due to hardening

$$df\left(\sigma_{ij},\, \varepsilon^p_{ij}\right) = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \boxed{\frac{\partial f}{\partial \varepsilon^p_{ij}} d\varepsilon^p_{ij}} = 0$$

\+ the flow law

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \qquad \Longrightarrow \qquad \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \underset{\uparrow}{\underline{\underline{d\lambda}}} \frac{\partial f}{\partial \varepsilon^p_{ij}} \frac{\partial g}{\partial \sigma_{ij}} = 0$$

the equation to determine the plastic multiplier

Constitutive models: Incremental plasticity – Flow rule – Hardening plasticity

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \underline{\underline{d\lambda}} \underbrace{\boxed{\frac{\partial f}{\partial \varepsilon_{ij}^{p}} \frac{\partial g}{\partial \sigma_{ij}}}}_{h} = 0$$

$$d\lambda = -\frac{1}{h} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$

$$\updownarrow$$

$$d\sigma_{ij} = C_{ijkl} \left(d\varepsilon_{kl} - d\lambda \frac{\partial g}{\partial \sigma_{kl}} \right)$$

$$d\lambda = -\frac{H_{kl}\, d\varepsilon_{kl}}{H - h}$$

$$\updownarrow$$

$$d\sigma_{ij} = \underbrace{\boxed{\left(C_{ijkl} - \frac{H_{kl}\, H_{kl}^{*}}{H - h} \right)}}_{C_{ijkl}^{ep}} d\varepsilon_{kl}$$

Constitutive models: Examination questionnaire

1. Do the constitutive laws depend on the reference configuration?

2. Do the constitutive laws give just a simple best fit to the experimental data? Give a short explanation to your answer.

3. The stress is characterized by a) stress vector, b) fourth order stress tensor or c) symmetric second order tensor

4. Consider 3D case. To define the stress state at a given body point it is necessary to know
   a) 9 different numbers, b) at least 6 different numbers, c) not more then 2 different numbers or
   d) not more than 6 different numbers

5. Does the trace of the stress tensor changes when rotating the coordinate system the tensor is defined to a certain angle?

6. Give the definition for deviator of a second order tensor. Prove that the trace of the deviator tensor is equal to zero.

7. Which process characteristics are involved in static analysis? Give the definition for statically admissible set.

8. Which process characteristics are involved in kinematic analysis? Give the definition for kinematically admissible sets.

Constitutive models: Examination questionnaire

9. Give the generalized Hooke”s law for a hyperelastic material. How many distinct components has the constitutive matrix in this case? Give explanation to your answer.

10. Consider the generalized Hooke”s law for isotropic linear elastic material. Give the general form of the isotropic elastic tensor. How many constants have to be determined in order to specify the isotropic stiffness matrix?

11. Which advantage the hyperelastic (Green-elastic) constitutive law has compared to the Cauchy elastic relationship? Give short explanation to your answer.

12. Give the stress-strain relationship in which tangent (tangential) stiffness tensor is involved. To which kind of elastic theories this relationship belongs?

13. Give the stress-strain relationship in which secant stiffness tensor is involved. To which kind of elastic theories this relationship belongs?.

14. Does the mean stress (or first stress invariant) influence the failure when applying von Mises failure criterion?

15. Is it possible with Drucker’s plasticity theory to describe non associated plastic behaviour?

16. Derive the specific form of the associated flow rule for von Mises

17. Derive the elastoplastic material stiffness tensor for associated von Mises plasticity