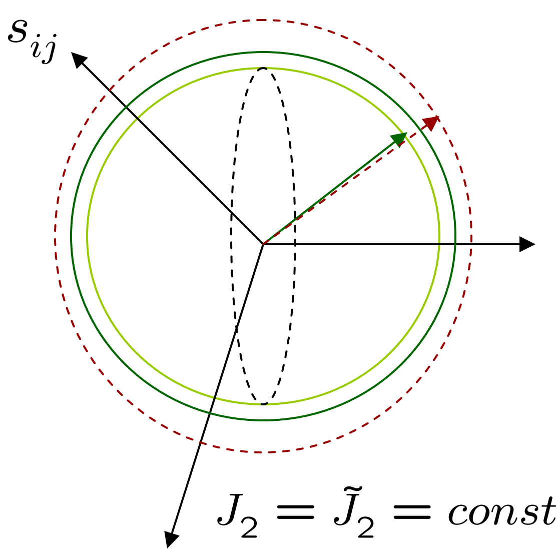

Active process:

$$ds_{ij} = 2\,G_s\,(J_2)\;de_{ij} + 2\,G'_s\,(J_2)\;e_{ij}\,dJ_2$$

Passive process:

$$ds_{ij} = 2\,G\,de_{ij}$$

If now

$dJ_2 \rightarrow 0$	neutral process

$$\Downarrow$$

$$\frac{(de_{ij})_{active}}{(de_{ij})_{passive}} = \frac{G}{G_s\,(J_2)}$$

*only active loading is implemented in numerical procedures

Consequently invoked

- Prager‘s consistency condition: two loading paths anyhow close to the neutral path must lead to equal resulting deformations.
- Prager‘s flow theory of plasticity

Prager's theory of plastic flow:

consistency condition + the following assumptions ->

1. the increment of the strain deviator is fully determined by the stress deviator and its increment;
2. the relation is **linear** regarding de_{ij} and ds_{ij} in both elastic and plastic cases;
3. the current yield (limit) surfaces are hyperspheres as in deformation plasticity theory;
4. Elastic and plastic strain increments are additive and their sum equals to the total strain increment (it is the total strain increment that is related to the velocities!)

$$\Downarrow \qquad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)!$$

definition

Loading (additional charge/discharge) from plastic case + 1 to 3 yield:

$$de_{ij} = \begin{cases} A_{ijkl}\, ds_{kl} & dJ_2 \geq 0 \\ ds_{ij}/2G & dJ_2 \leq 0 \end{cases}$$

Prager's flow theory ->

$$de_{ij} = \begin{cases} \overset{?}{\boxed{A_{ijkl}}}\, ds_{kl} & dJ_2 \geq 0 \\ ds_{ij}/2G & dJ_2 \leq 0 \end{cases}$$

+

elastic and plastic strains additivity

$$\Downarrow$$

$$de^p_{ij} = \begin{cases} \overbrace{\left(A_{ijkl} - \frac{1}{2G}\delta_{ik}\delta_{jl}\right)}^{D^p_{ijkl}} ds_{kl} & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases}$$

$$D^p_{ijkl} = A_{ijkl} - \frac{1}{2G}\delta_{ik}\delta_{jl}$$ plastic compliance matrix

Prager's flow theory ->

$$de_{ij} = \begin{cases} A_{ijkl}\, ds_{kl} & dJ_2 \geq 0 \\ ds_{ij}/2G & dJ_2 \leq 0 \end{cases}$$

+

elastic and plastic strains additivity

$$\Downarrow$$

$$de^p_{ij} = \begin{cases} \left(A_{ijkl} - \frac{1}{2G}\delta_{ik}\delta_{jl}\right) ds_{kl} & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases}$$

? $D^p_{ijkl} = A_{ijkl} - \frac{1}{2G}\delta_{ik}\delta_{jl}$

+

consistency condition

$$de^p_{ij} = \begin{cases} D_{ij} s_{kl}\, ds_{kl} & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases} \quad \underset{*}{\Rightarrow} \quad de^p_{ij} = \begin{cases} \alpha_2\left(J_2\right)\, s_{ij} dJ_2 & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases}$$

Prager‘s flow theory ->

Consistency condition: two loading paths anyhow close to the neutral path must lead to equal resulting deformations :

$$dJ_2 = 0 \quad \left(\Leftrightarrow \quad s_{ij} ds_{ij} = 0\right) \qquad \Rightarrow \quad d\,e^p_{ij} = 0 \quad \left(\Leftrightarrow \quad D^p_{ijkl} ds_{kl} = 0\right)$$

D_{ij} - an arbitrary tensor function of the stress deviator

$$\Downarrow$$

$$D^p_{ijkl} = D_{ij}\, s_{kl}$$

$$\Downarrow$$

$$de^p_{ij} = \begin{cases} D_{ij} s_{kl}\, ds_{kl} & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases} \quad \underset{*}{\Rightarrow} \quad de^p_{ij} = \begin{cases} \alpha_2 \left(J_2\right)\, s_{ij} dJ_2 & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases}$$

Prager's flow theory ->

$$de^p_{ij} = \begin{cases} D_{ij} s_{kl}\, ds_{kl} & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases} \quad \overset{}{\underset{*}{\Rightarrow}} \quad de^p_{ij} = \begin{cases} \alpha_2 \left(J_2\right)\; s_{ij} dJ_2 & dJ_2 \geq 0 \\ 0 & dJ_2 \leq 0 \end{cases}$$

* $D_{ij}\left(\boldsymbol{s}\right)$ + Cayley – Hamilton theorem + assumptions

- no influence of the third dev. stress invariant
- linear terms only

$$\Downarrow$$

$$\boldsymbol{D}\left(\boldsymbol{s}\right) = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{s} + \alpha_3 \boldsymbol{s}^2$$

$$D_{ij}\left(\boldsymbol{s}\right) = \cancel{\underline{\alpha_1 \delta_{ij}}} + \alpha_2 s_{ij} + \cancel{\underline{\alpha_3 s_{ik} s_{kj}}}$$

de^p_{ij} is deviator $\Rightarrow$ D_{ij} is also deviator $\Leftrightarrow$ $D_{ij}\delta_{ij} = 0$

$$\Downarrow$$

$$\alpha_1 = 0$$

$$\boxed{f\left(s_{ij},\, e^p_{ij},\, \chi^p_k\right)}$$

χ^p_k - parameters, characterizing loading history (internal state parameters like damage)

$\chi^p_k\,,\;\; e^p_{ij}$ - do not change during passive paths (also during unloading)

Examples of hardening measures:

Odqvist parameter proportional to the length of the plastic deformation trajectory arc :

$$\chi^p = \int \sqrt{2\, de^p_{ij}\, de^p_{ij}}$$

Plastic work

$$W^p = \int \sigma_{ij}\, de^p_{ij}$$

$$f\left(s_{ij},\, \underline{e^p_{ij}},\, \chi^p_k\right) \longrightarrow f^{failure}\left(s_{ij}\right)$$

Examples:

Isotropic hardening :

$$F(s_{ij}) = \kappa(\chi^p) \qquad f(s_{ij},\, \chi^p) = F(s_{ij}) - \kappa(\chi^p)$$

Kinematic (anisotropic) hardening (ideal Bauschinger effect) :

$$F(s_{ij} - \alpha_{ij}) = \kappa_p \qquad f(s_{ij},\, \chi^p) = F(s_{ij} - \alpha_{ij}) - \kappa_p$$

$$\alpha_{ij} = c\, e^p_{ij}$$

Mixed kinematic and isotropic hardening :

$$F(s_{ij} - \alpha_{ij}) = \kappa_p(\chi^p)$$

$$f(s_{ij},\, \chi^p) = F(s_{ij} - \alpha_{ij}) - \kappa_p(\chi^p)$$

$$f\left(s_{ij},\, e^{p}_{ij},\, \chi^{p}_{k}\right)$$ loading function

The process is called **loading** if both elastic and plastic strains change

$$f\left(s_{ij},\, e^{p}_{ij},\, \chi^{p}_{k}\right) \;=\; 0$$

equation of the
yield-limit-failure surface
(metal plasticity – deviatoric 5D space)

Consistency condition:

small additional charge gives small change in the yield surface

$$f(\sigma_{ij},\, \varepsilon^{p}_{ij},\, \chi^{p}_{k}) = 0$$

$$f(\sigma_{ij} + d\sigma_{ij},\, \varepsilon^{p}_{ij} + d\varepsilon^{p}_{ij},\, \chi^{p}_{k} + d\chi^{p}_{k}) = 0$$

small increment -> current state belongs to *f*

$$df = 0$$

$$f(s_{ij},\, e^{p}_{ij},\, \chi^{p}_{k}) = 0 \longrightarrow \frac{\partial f}{\partial s_{ij}}\, ds_{ij} + \frac{\partial f}{\partial e^{p}_{ij}}\, de^{p}_{ij} + \frac{\partial f}{\partial \chi^{p}_{k}}\, d\chi^{p}_{k} = 0$$

$$f\left(s_{ij},\, e_{ij}^{p},\, \chi_k^{p}\right)$$ loading function

The process is called **loading** if both elastic and plastic strains change

$$f\left(s_{ij},\, e_{ij}^{p},\, \chi_k^{p}\right) = 0$$ equation of the yield-limit-failure surface (metal plasticity)

Consistency condition:

small additional charge gives small change in the yield surface

$$\frac{\partial f}{\partial s_{ij}}\, ds_{ij} + \frac{\partial f}{\partial e_{ij}^{p}}\, de_{ij}^{p} + \frac{\partial f}{\partial \chi_k^{p}}\, d\chi_k^{p} = 0 \qquad (1)$$

\+ zero plastic deformation during neutral stress paths

$$\left.\begin{aligned} d\, e_{ij}^{p} &= 0 \quad \left(\Leftrightarrow \;\; D_{ijkl}^{p}\, ds_{kl} = 0\right) \\ d\chi_k^{p} &= 0 \end{aligned}\right\} \longleftarrow \boxed{\frac{\partial f}{\partial s_{ij}}\, ds_{ij} = 0} \qquad (2)$$

condition for neutral paths

$$d\,e^p_{ij} = 0 \;\left(\Leftrightarrow\; D^p_{ijkl} ds_{kl} = 0\right) \longleftarrow \frac{\partial f}{\partial s_{ij}} ds_{ij} = 0 \rightarrow D_{ij} \frac{\partial f}{\partial s_{kl}} ds_{kl} = 0 \qquad (2)$$

$$\Downarrow$$

$$\left(D^p_{ijkl} - D_{ij}\frac{\partial f}{\partial s_{kl}}\right) ds_{kl} = 0$$

$$\Downarrow$$

$$de^p_{ij} = \begin{cases} D_{ij}\dfrac{\partial f}{\partial s_{kl}} ds_{kl} & \dfrac{\partial f}{\partial s_{kl}} ds_{kl} > 0 \\[2ex] 0 & \dfrac{\partial f}{\partial s_{kl}} ds_{kl} \le 0 \end{cases}$$

loading-unloading conditions
in terms of the loading function

$\Downarrow$ Consistency condition (1)

$$D_{ij}\frac{\partial f}{\partial e^p_{ij}} = -1 - \frac{\dfrac{\partial f}{\partial \chi^p_k} d\chi^p_k}{\dfrac{\partial f}{\partial s_{ij}} ds_{ij}}$$

for hardening plasticity

One equation for the 5 unknown components of the deviatoric tensor ***D***

$$\boxed{\oint_{\sigma}\left(\sigma_{ij}-\sigma_{ij}^{0}\right)d\varepsilon_{ij}\geq 0}$$

According to this postulate, assuming isothermal conditions,

the work done by an additional load on the body during a closed cycle of

application-and-removal of the added load is non-negative

$$\Downarrow$$

and this results

in a special system of constitutive equations for the plastic range which

is certainly of importance in the application of the theory to metals.

Constitutive models: Incremental plasticity – Drücker's postulate

$$\boxed{\oint_\sigma \left(\sigma_{ij} - \sigma_{ij}^0\right) d\varepsilon_{ij} \geq 0}$$

the elastic work over closed path =0 $\Downarrow$

$$\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0\right) d\boldsymbol{\varepsilon}^p \geq 0 \quad \xrightarrow[\text{is arbitrary}]{\sigma_{ij}^0} \quad d\boldsymbol{\sigma}\, d\boldsymbol{\varepsilon}^p \geq 0$$

$$\underbrace{\hphantom{\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0\right) d\boldsymbol{\varepsilon}^p \geq 0 \qquad d\boldsymbol{\sigma}\, d\boldsymbol{\varepsilon}^p \geq 0}}_{}$$

basic inequalities of Drucker's elastoplasticity

$\Downarrow$

the limit surface is not concave + the increment of the plastic strain coincides with the normal direction to the limit surface

$\Downarrow$

$$\boxed{d\boldsymbol{\varepsilon}^p \sim \mathrm{grad}\, f}$$

$\Downarrow$

$$D_{ij} \sim \mathrm{grad}\, f$$

$$d\boldsymbol{\varepsilon}^p \sim \text{grad } f \qquad \Rightarrow \qquad D_{ij} \sim \text{grad } f$$

$$\text{grad } f \longrightarrow \frac{\partial f}{\partial s_{ij}} \qquad \Rightarrow \qquad \boxed{D_{ij} = k \frac{\partial f}{\partial s_{ij}}}$$

$$de_{ij}^p = \begin{cases} D_{ij} \dfrac{\partial f}{\partial s_{kl}} ds_{kl} & \dfrac{\partial f}{\partial s_{kl}} ds_{kl} > 0 \\ 0 & \dfrac{\partial f}{\partial s_{kl}} ds_{kl} \leq 0 \end{cases}$$

$$\Downarrow$$

$$de_{ij}^p = k \frac{\partial f}{\partial s_{ij}} \frac{\partial f}{\partial s_{kl}} ds_{kl} \qquad \text{if} \qquad \frac{\partial f}{\partial s_{mn}} ds_{mn} > 0$$

Constitutive models: Incremental plasticity – Drücker's postulate

$$\boxed{d\boldsymbol{\varepsilon}^p \sim \mathrm{grad}\, f} \quad \Rightarrow \quad D_{ij} \sim \mathrm{grad}\, f$$

$$\Downarrow$$

$$D_{ij} = k\,\frac{\partial f}{\partial s_{ij}}$$

$$\Downarrow$$

$$de^p_{ij} = k\,\frac{\partial f}{\partial s_{ij}}\frac{\partial f}{\partial s_{kl}}\,ds_{kl} \qquad \text{if} \qquad \frac{\partial f}{\partial s_{mn}}\,ds_{mn} > 0$$

$$\Downarrow$$

$$D_{ij}\frac{\partial f}{\partial e^p_{ij}} = -1 - \frac{\dfrac{\partial f}{\partial \chi^p_k}\,d\chi^p_k}{\dfrac{\partial f}{\partial s_{ij}}\,ds_{ij}} \xrightarrow{\dfrac{\partial f}{\partial e^p_{mn}} \neq 0} k = -\frac{\dfrac{\partial f}{\partial s_{kl}}\,ds_{kl} + \dfrac{\partial f}{\partial \chi^p_k}\,d\chi^p_k}{\dfrac{\partial f}{\partial e^p_{ij}}\dfrac{\partial f}{\partial s_{ij}}\dfrac{\partial f}{\partial s_{kl}}\,ds_{kl}}$$

consistency condition

associated plastic law, associated plasticity -
plastic flow law associated with the limit (loading) surface
Prager's assumptions + Drucker's postulate