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THE WEDGE DISSIPATION EFFECTS IN SURFACE GRAVITY WAVES IN A CHANNEL*

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We study the dynamics of two-dimensional surface gravity waves of fluid in a channel. The viscous dissipation in the bulk of the fluid is assumed to be negligible by comparing with the dissipation in the vicinity of the contact line. Capillarity effects are not considered. Dissipation in the vicinity of the contact line is described by contact line dissipation model suggested in de Ruijter et al. (*Langmuir* **15** (1999), 2209) who introduces a phenomenological dissipation term proportional to the contact line length. We obtain numerically the time evolution of the fluid interfase. We compare dynamics of the waves, triple contact line and contact angle with solution for standing waves.

1. Introduction. The description of the contact line motion is still a subject of active research in spite of a large number of articles published in the best scientific journals each year. Conventional hydrodynamic approaches use the disparate boundary conditions at the triple contact line – free movement of the contact line (in ideal liquid model), movement with fixed three phase contact angle (in ideal liquid model when added capillarity effects), stick-condition (in viscous liquid model) and give nonrealistic description of the dissipation (zero or infinite large) of the contact line. It is not conventional approach to modify these models to describe large but finite contact line dissipation. It is of actuality to analyze and compare different approaches and to add the finite dissipation effects in these approaches for various fluid systems.

2. Formulation. Our goal here is to test the phenomenological contact line dissipation model [1]–[4] for 2D surface gravity waves of fluid in a rectangular channel. Dissipation along the contact line is described by dissipation function T (per unit length of the contact line)

$$T = \xi v^2, \tag{1}$$

where v is the contact line velocity. Only one parameter ξ (that we call the dissipation coefficient) is necessary to describe the wedge dissipation. It depends on the three phase system and according to various experimental data $\xi \gg$ shear viscosity, it ranges from 30 [5] to 10^7 [6].

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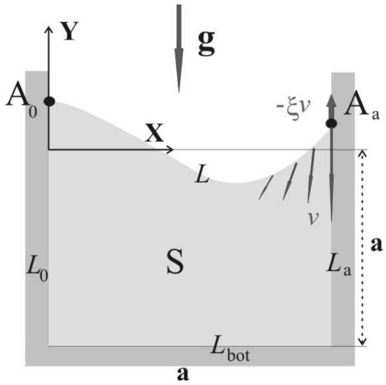

Fig. 1. Definition sketch

We consider the 2D motion of an incompressible liquid in a rectangular homogeneous solid channel with vertical walls under the action of the gravity g (see Fig. 1). The Cartesian coordinate system (x, y) depicted in Fig. 1 is employed. The distance between the walls L_0, L_a is a; L_{bot} is the channel bottom; the depth is a. We denote by S the liquid domain, by $L \equiv \left\{\mathbf{R}^L = \left(R_x^L, R_y^L\right)\right\}$ – the liquid interface with the boundaries $\mathbf{A}_0$ and $\mathbf{A}_a$. All lengths are normalized by a, time by $\sqrt{a/g}$, velocities by $\sqrt{ag}$, acceleration by g. We are interested in the small-amplitude wave motion of the interface, so we can neglect the fluid viscosity and assume that the energy is dissipated only at the contact line. The liquid flow in $S = \{R\,(x,y)\}$ is assumed to be irrotational, therefore it can be described in terms of the velocity potential φ: $v = \operatorname{grad}\varphi$. We use dimensionless $\xi = \xi\sqrt{a/g}/\rho$ and the renormalized $\varphi = \varphi/a\sqrt{ag}$. φ must satisfy the Laplace equation in the domain S:

$$\nabla^2\varphi\,(R,t) = 0, \quad R(x,y) \in S. \tag{2}$$

The container bottom and walls are rigid and impermeable, therefore,

$$\partial\varphi\,(R)/\partial x = 0, \quad R \in \{L_0, L_a\}\,,\ \partial\varphi\,(R)/\partial y = 0, \quad R \in L_{bot}. \tag{3}$$

The dynamic boundary condition on L is based on the Bernoulli equation and is given by

$$\partial\varphi\left(\mathbf{R}^L\right)/\partial t = -v\left(\mathbf{R}^L\right)^2\Big/2 - R_y^L;\ \mathbf{R}^L \in L/\{\mathbf{A}_0 \cup \mathbf{A}_a\}, \tag{4}$$

in inner point of L and in boundary points

$$\partial\varphi\left(\mathbf{R}^L\right)/\partial t = -v\left(\mathbf{R}^L\right)^2\Big/2 - R_y^L - \xi\left(R_x^L\right)\varphi, \quad \mathbf{R}^L \in \mathbf{A}_0 \cup \mathbf{A}_a. \tag{5}$$

As initial position of the free surface and the initial distribution of the potential we take

$$\begin{aligned} \varphi\,(x,y,0) &= \left(\mu + 5\mu^3/32\right)\cos 2\pi x \exp(2\pi y)\Big/(2\pi)^{3/2} + o\left(\mu^3\right), \\ L(x,0) &= 0 + o\left(\mu^3\right) \end{aligned} \tag{6}$$

i.e. the same as for the asymptotical solution [7] in a power series for small parameter $\mu \ll 1$ ($\mu \approx$ amplitude of the wave) for standing waves of finite amplitude in a channel:

$$L(x,t) = \left(\left(\mu + 3\mu^3/32\right)\sin\omega t + \mu^3 \sin 3\omega t/16\right)\cos 2\pi x/2\pi + \mu^2\left(1-\cos 2\omega t\right)\cdot$$

$$\cdot \cos 4\pi x/8\pi + \left(9\mu^3 \sin\omega t - 3\mu^3 \sin 3\omega t\right)\cos 6\pi x/64\pi + o\left(\mu^3\right); \quad (7)$$

$$\omega = \sqrt{2\pi\left(1-\mu^2/4 - 13\mu^4/128\right)}$$

We solve (2)–(5) numerically using the method, described in [8].

3. Numerical results. When a surface tension is not considered, there are no limitations for the magnitude of the contact angle at the border point, but in standing waves contact angle is always 90°. According to the asymptotic solution (6), (7) border point moves sinusoidaly. In numerical calculations for given μ we take the same initial conditions (7) as in the asymptotic solution and investigate changes of the waves, contact line and contact angle caused by dissipation ξ.

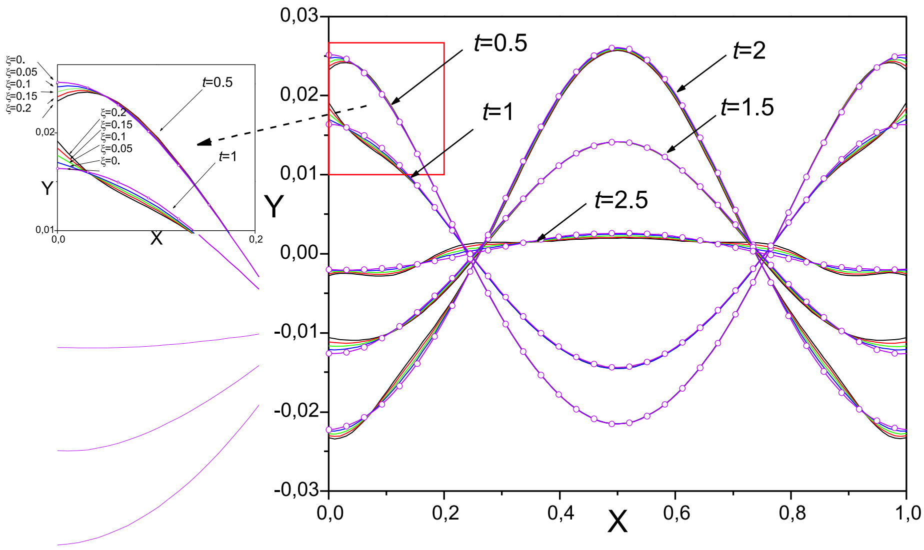

Fig. 2. The free line L as a function of the distance to the wall x at moments of time $t = 0.5$, 1, 1.5, 2, 2.5 in cases $\xi = 0$ (standing waves) – with circles, $\xi = 0.05$, 0.1, 0.15, 0.2 ($\mu = 0.157$)

First, we fix the amplitude of the standing waves $\mu = 0.157$ and then we study the influence of the magnitude of the dissipation ξ. In Fig. 2 we show the numerically obtained solutions for the free line L at dimensionless time moments $t = 0.5$, 1, 1.5, 2, 2.5 in the first period of oscillations for $\xi = 0$, 0.05, 0.1, 0.15, 0.2. The numerical solution for the standing waves when the dissipation $\xi = 0$, is shown with empty circles. As it can bee seen from Fig. 2, smooth change of dissipation ξ leads to a smooth change of the fluctuations of the free line L. The solution for the free line L when ξ decreases to zero, approaches uniformly to the solution for a standing wave when there is no dissipation at the wall. The main difference with standing waves in the first period of oscillations is at border regions. In the next periods of oscillations the falling behind of the border points the fluid interface L leads to a bigger and bigger changes of the whole free surface. Similarly, when ξ decreases, the dynamical contact angle converges uniformly to $90°$. In Fig. 3 we show the contact angle at the border point $\mathbf{A}_0$ of numerically obtained line L in the time interval $t \in [0, 8]$ for different values of the dissipation $\xi = 0$, 0.05, 0.1, 0.15, 0.2. It is seen that when ξ decreases, the dynamic contact angle converges uniformly to $90°$.

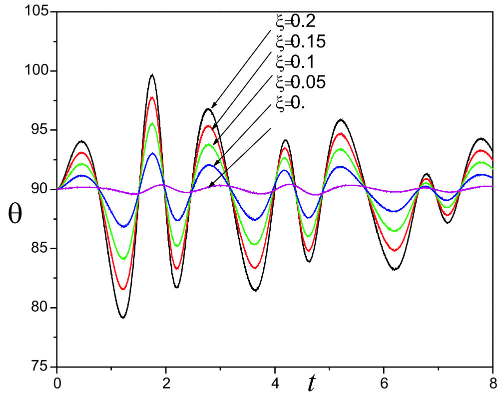

Fig. 3. The contact angle as a function of the dimensionless time at the border point A_0

We obtain that the dynamics of the border points is not sinusoidal as for standing waves, but it is complicate a function of the flow field. So is for dynamic contact angles. This fact is illustrated for the first period of oscillation in Fig. 4 for $\xi = 0.2$. The change of the height with time of the border point $A_0(t)$ for $\xi = 0.2$ is shown in the figure with thick

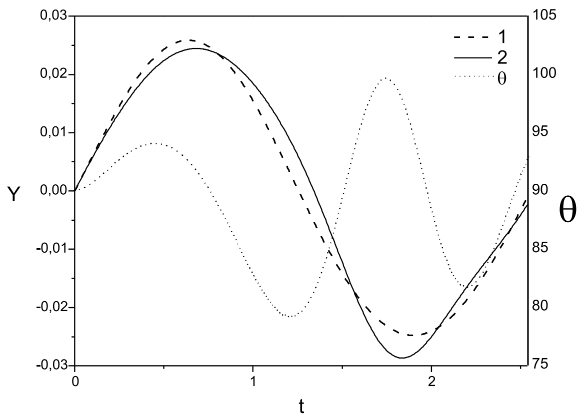

Fig. 4. The change of the height with time of the border point for $\xi = 0.2$ - with thick solid line; for $\xi = 0$ – with thick dashed line. The contact angle as a function of the time at the border point A_0 – with solid line

solid line and for $\xi = 0$ (standing wave) – with thick dotted line. In the same figure the contact angle for $\xi = 0.2$ is shown with thin solid line. The presence of the dissipation ξ leads to slowing down of the motion of the border point in the first period of oscillation at almost all time moments except in time interval [2.03–2.44]. The change in this time interval is determined by the appearance of velocity at the contact points bigger than at the neighboring parts of the free line.

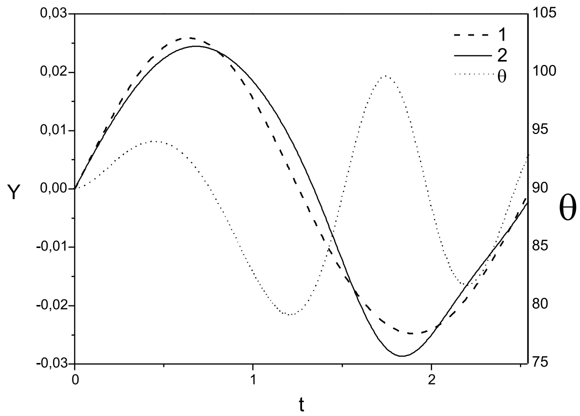

Fig. 5. The contact angle as a function of the dimensionless time at the border point A_0 in cases $\mu = 0.157$, 0.125, 0.094, 0.063, 0.031; $t = [0, 8]$

Now, we study the influence of the amplitude μ on the contact angle oscillations for given $\xi = 0.2$. In Fig. 5 are shown the solutions for the contact angle at point $\mathbf{A}_0$ for amplitudes $\mu = 0.157, 0.125, 0.094, 0.063, 0.031$. It is seen that the contact angle changes smoothly when we change smoothly the amplitude μ. As $\mu \to 0$, so does the deviation of the contact angle from $90°$.

4. Conclusions. Numerical results for several cases, from which we could make a conclusion about quality specifications of the theoretical model were obtained. We have obtained, as well, that the contact angle behavior at the border points and the free line could be described by friction coefficient. In our study we found that the friction at the border points and free line lead to a qualitative change of the shape of the free surface and the three phase dynamic contact angle. The obtained results show that with the suggested numerical algorithm we can effectively solve the Laplace equation with mixed boundary condition in non orthogonal domain and can study the behavior of the fluid in a vessel when friction forces at the contact line are present.

REFERENCES

[1] P. G. De Gennes. Wetting: statics and dynamics. *Rev. Mod. Phys.* **57** (1985), 827–863.

[2] M. J. de Ruijter, J. de Coninck, G.Oshanin. Droplet Spreading: Partial Wetting Regime Revisited. *Langmuir* **15** (1999), 2209–2216.

[3] V. Nikolayev, D. Beysens. Relaxation of non-spherical sessile drops towards equilibrium. *Phys. Rev. E*, **65** (2002), 046135.

[4] S. Iliev, N. Pesheva, V.S. Nikolayev. Quasistatic relaxation of arbitrarily shaped sessile drops. *Phys. Rev. E*, **72** (2005), 011606.

[5] S. Iliev, N. Pesheva, V. S. Nikolayev. Dynamic modelling of the deformed contact line under partial wetting conditions: quasi-static approach. Proc. European Coating Symposium 2007, Paris, 12-14/9 2007, 121–124; *Europ. Phys. J. Spec. Topics*, submitted.
[6] C. Andrieu, D. A. Beysens, V. S. Nikolayev, Y. Pomeau. Coalescence of sessile drops. *J. Fluid Mech.* **453** (2002), 427–438.
[7] W. G. Penney, A. T. Price. Some gravity wave problems in the motion of perfect liquids. Part II. Finite periodic stationary gravity waves in a perfect liquid. *Philos. Trans. R. Soc. London*, Ser. A. **244** (1952), 254–284.
[8] D. Iliev. Local variation method to calculate two dimensional gravity waves in container. *Mathematics and Education in Mathematics*, **36** (2007), 240–245.

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ЕФЕКТИ НА ДИСИПАЦИЯТА ОТ ОБЛАСТТА НА ТРИФАЗЕН КОНТАКТ ВЪРХУ ПОВЪРХНОСТНИТЕ ГРАВИТАЦИОННИ ВЪЛНИ НА ТЕЧНОСТ В КАНАЛ

Димитър Станимиров Илиев, Станимир Димитров Илиев

В статията се изследва динамиката на двумерни повърхностни гравитационни вълни на флуид в канал. Вискозната дисипация във вътрешността на флуидната среда се предполага пренебрежимо малка спрямо дисипацията в околността на контактната линия. Капилярни ефекти не се разглеждат. Дисипацията в околността на контактната линия се описва чрез модела на “дисипация на контактната линия” предложен от de Ruijter et al. (*Langmuir* **15** (1999), 2209), в който се въвежда феноменологичен дисипативен член, пропорционален на дължината на контактната линия. Получена е числено еволюцията на повърхността на флуидната среда. Сравнява се получената динамика на вълните, на контактните линии и контактни ъгли с тези на стояща вълна.