Constitutive models: Geometrical interpretation of stress invariants

Deviatoric plane equation:

\[ \sigma_1 + \sigma_2 + \sigma_3 = 3p \quad \Rightarrow \quad N \in \Pi \]

\[ P \in \Pi \quad \text{+ the deviatoric plane is perpendicular to the hydrostatic axis} \]
Constitutive models: Geometrical interpretation of stress invariants

\[ P(\sigma_1, \sigma_2, \sigma_3) \quad N(p, p, p) \]

\[ e'_1 = \frac{1}{\sqrt{6}} (2, -1, -1) \]

\[ \overrightarrow{NP} \cdot \overrightarrow{e'_1} = ||\overrightarrow{NP}|| \cos \Psi = \sqrt{2J_2} \cos \Psi \]

\[ \Rightarrow \cos \Psi = \frac{\overrightarrow{NP} \cdot \overrightarrow{e'_1}}{\sqrt{2J_2}} = \frac{1}{\sqrt{2J_2}} \frac{1}{\sqrt{6}} (2s_1 - s_2 - s_3) \]

with \( -s_2 - s_3 = s_1 \)

\[ \cos \Psi = \frac{\sqrt{3}}{2} \frac{s_1}{\sqrt{2J_2}} \quad \Psi \in [0, \pi/3] \]

\[ \cos 3\Psi = 4 \cos^3 \Psi - 3 \cos \Psi \quad \Rightarrow \]

\[ \cos 3\Psi = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2}^3} \]

Lode's angle
Constitutive models: Geometrical interpretation of stress invariants

\[
P(\sigma_1, \sigma_2, \sigma_3) \quad N(p, p, p)
\]

\[
e_1' = \frac{1}{\sqrt{6}} (2, -1, -1)
\]

\[
\overrightarrow{NP} \cdot \overrightarrow{e_1'} = \|\overrightarrow{NP}\| \cos \Psi = \sqrt{2J_2} \cos \Psi
\]

\[
\Rightarrow \cos \Psi = \frac{\overrightarrow{NP} \cdot \overrightarrow{e_1'}}{\sqrt{2J_2}} = \frac{1}{\sqrt{2J_2}} \frac{1}{\sqrt{6}} (2s_1 - s_2 - s_3)
\]

with \(-s_2 - s_3 = s_1\)

\[
\cos \Psi = \frac{\sqrt{3}}{2} \frac{s_1}{\sqrt{2J_2}} \quad \Psi \in [0, \pi/3]
\]

\[
\cos 3\Psi = 4 \cos^3 \Psi - 3 \cos \Psi \quad \Rightarrow
\]

\[
\cos 3\Psi = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}}
\]

Lode’s angle
Constitutive models: Failure analysis

Complete progressive analysis of the structure up to collapse

- concrete reactor vessels;
- nuclear containment structures;
- parts of offshore platforms ...  

Experimental study is very expensive and therefore modelling is of importance.

Most commonly used material laws consist of:

- theory of elasticity + criteria defining failure

\[
\begin{align*}
&f(\sigma_{ij}, T, t, \varphi) \\
&f(\sigma_{ij}) \\
&f(\sigma_1, \sigma_2, \sigma_3) \\
&f(I_1, J_2, J_3) \\
&f(\sqrt{3}p, \rho, \psi), \quad \rho = \sqrt{2J_2}
\end{align*}
\]
Constitutive models: Failure Surface

\[ f(\sigma_{ij}) \quad f(\sigma_1, \sigma_2, \sigma_3) \]
\[ f(I_1, J_2, J_3) \]
\[ f(\sqrt{3}p, \rho, \Psi) \quad \rho = \sqrt{2J_2} \]

Geometrical interpretation:

- **Failure curve** – the cross section of the failure surface and a deviatoric plane

\[ p = \text{const} \]

- **Meridians** of the failure surface – the intersection curves between the failure surface and a plane (meridian plane) containing the hydrostatic axis

\[ \Psi = \text{const} \]
Constitutive models: Failure Surface

Failure curve (FC) characteristics (based on experimental results):

\[ f \left( \sqrt{3} p, \rho, \psi \right), \quad \rho = \sqrt{2 I_2} \]
\[ p = \text{const} \]

- The FC is smooth.
- The FC is convex:

\[ \frac{\partial^2 \rho}{\partial \psi^2} < \rho + \frac{2}{\rho} \left( \frac{\partial \rho}{\partial \psi} \right) \]

- The FC is nearly triangular for small hydrostatic pressures and becomes more circular with increasing \( p \).

(FC is nonaffine)
Constitutive models: Failure Surface

\[ f\left(\sqrt{3}p, \rho, \Psi\right) = 0, \quad \rho = \sqrt{2J_2} \]

\[ \Psi = \text{const}; \quad \Psi = 0, \quad \Psi = \pi/3 \quad \text{meridian plane} \]

**Compressive meridian:**
\[ \sigma_1 = \sigma_2 > \sigma_3 \]
\[ \downarrow \]
\[ \Psi = \pi/3 \]

triaxial compression: hydrostatic pressure + axial compression

**Tensile meridian:**
\[ \sigma_3 = \sigma_2 < \sigma_1 \]
\[ \downarrow \]
\[ \Psi = 0 \]

triaxial tension: hydrostatic pressure + axial tension

**Shear meridian:**
\[ \sigma_1, \frac{\sigma_1 + \sigma_3}{2}, \sigma_3 \]
\[ \Psi = \pi/6 \]
Constitutive models: Failure Surface

Copmressive meridian: one dimentional compression

\[ \sigma_3 = -\sigma \quad \sigma_1 = \sigma_2 = 0 \]

\[ 3p = \sigma_{ij} \delta_{ij} = -\sigma \]

\[ s_1 = \sigma_1 - p = \frac{\sigma}{3} \quad s_2 = s_1 \quad s_3 = \sigma_3 - p = -\frac{2}{3} \sigma \]

\[ J_2 = \frac{1}{2} s_{ij} s_{ij} \quad \rightarrow \quad J_2 = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2) = \frac{1}{3} \sigma^2 \]

\[ J_3 = \frac{1}{3} s_{ij} s_{jkl} s_{kl} \quad \rightarrow \quad J_3 = \frac{1}{3} (s_1^3 + s_2^3 + s_3^3) = -\frac{2}{27} \sigma^3 \]

\[ \cos 3\Psi = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}} \quad \rightarrow \quad \cos 3\Psi = -\frac{3\sqrt{3}}{2} \frac{2}{27} \sigma^3 \frac{\sqrt{3^3}}{\sigma^3} = -1 \]

\[ 3\Psi = \pi \]

\[ P(\sqrt{3}p, \rho, \Psi) \quad \rightarrow \quad P(-\frac{\sqrt{3}}{3} \sigma, \sqrt{\frac{2}{3}} \sigma, \frac{\pi}{3}) \]

1D compression
Constitutive models: Failure Surface

Failure surface characteristics in the meridian planes

\[ f \left( \sqrt{3}p, \rho, \Psi \right), \quad \rho = \sqrt{2I_2} \]
\[ \Psi = \text{const} \]

- The FC depends on the hydrostatic component of stress, \( I_1 \).
- The meridians are curved, smooth and convex
- Tensile meridian is posed below the compressive one

- The tensile and compressive meridians tend to coincide with increasing the hydrostatic pressure (the ratio starting with 0.5 near the \( \pi \)-plane)

- Pure hydrostatic loading cannot cause failure
Constitutive models: Failure Models

\[ f(\sigma_{ij}) = 0 \]

Historical overview

**Hypothesis 1:** the theory of maximum normal stress states: the material strength is completely determined by the stress and the two strength limits in tension, \( \sigma_t \), and compression, \( \sigma_c \).

Galilei, Leibniz (17c)

\[
\begin{align*}
\sigma_3 &= -\sigma_c \\
\sigma_1 &\leq \sigma_t \\
\sigma_3 &\geq -\sigma_c \\
\sigma_1 &= \sigma_t
\end{align*}
\]

Bauschinger experiments 1874 on steel with different carbon content -> the limit of elastic response in torsion is appr. the half of the elastic limit in tension

**Hypothesis 2:** theory of maximum linear relative deformation (assess material strength through max tensile elongation (te))

\[
\begin{align*}
\sigma_1 - k(\sigma_2 + \sigma_3) &\leq \sigma_{te} \\
\sigma_3 - k(\sigma_1 + \sigma_2) &\geq -\sigma_c
\end{align*}
\]

Coefficient of lateral compression deformation:

\[
k = \frac{1}{4}
\]

\[ \Rightarrow \text{strength in pure compression exceeds 4 times the strength in tension} \]
Constitutive models: Failure Models

\[ f(\sigma_{ij}) = 0 \]

**Historical overview**

**Hypothesis 3:** the theory of maximum difference in normal stress in which the material strength is defined by the maximum shear stress; only maximum and minimum principle stresses play a role in this theory.

**Coulomb 1773**

\[ \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) \leq \frac{\sigma_{te}}{2} \]

Pure shear:

\[ \tau_{max} = \sigma_1 = -\sigma_3 \leq \frac{\sigma_{te}}{2} \]

(confirmed by the Bauschinger experiment)

**Duguet 1882**

assumed like Coulomb that the failure is due to shear; resistance to shear failure depends on cohesion and internal friction. The later value changes with change of the normal stress, \( \sigma_\nu \), acting on the shear plane.

\[ \tau + f \sigma_\nu = const \]

\[ f = 0.176 \]

coefficient of internal friction
Constitutive models: Failure Models

\[ f(\sigma_{ij}) = 0 \]

Historical overview

Mohr 1882; 1900; Mohr’s principle circles \( \rightarrow \) Mohr’s parabola as a hull for failure states

- **pc**: \( \sigma_2 = \sigma_1 = 0 \) \( \sigma_3 = -\sigma_c \)
- **ps**: \( \sigma_2 = 0 \) \( \sigma_1 = -\sigma_3 = \sigma_s \)
- **pte**: \( \sigma_3 = \sigma_2 = 0 \) \( \sigma_1 = \sigma_{te} \)