Principal strains, principal directions

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{pmatrix}$$

The principal strains are determined from the characteristic (eigenvalue) equation:

$$|\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij}| = 0 \qquad k = 1, 2, 3$$

The three eigenvalues are the principal strains. The corresponding eigenvectors designate the direction (principal direction) associated with each of the principal strains:

$$(\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij})n_i^{(k)} = 0$$

! In general the principal directions for the stress and the strain tensors do not coincide.

Strain invariants:

$$|\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij}| =$$

$$\left(\varepsilon^{(k)}\right)^3 - I_1^\varepsilon \left(\varepsilon^{(k)}\right)^2 + I_2^\varepsilon \left(\varepsilon^{(k)}\right) - I_3^\varepsilon = 0$$

$$I_1^\varepsilon = \varepsilon_{ii} = tr\boldsymbol{\varepsilon}$$

$$I_2^\varepsilon = \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}\right)$$

$$I_3^\varepsilon = \det\boldsymbol{\varepsilon}$$

$$I_1^\varepsilon = tr\boldsymbol{\varepsilon}$$

$$I_2^\varepsilon = tr\boldsymbol{\varepsilon}^2$$

$$I_3^\varepsilon = tr\boldsymbol{\varepsilon}^3$$

Note: $\nabla \cdot \mathbf{u} = u_{i,i} = \varepsilon_{ii}$

Decomposition to spherical (hydrostatic) and deviatoric parts

$$\varepsilon_{ij} = e_{ij} + \varepsilon_M \delta_{ij}$$

$$\boldsymbol{\varepsilon}_M = \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix} \qquad \varepsilon_M = \frac{\varepsilon_{ii}}{3}$$

spherical or hydrostatic part

$$\boldsymbol{\varepsilon}_D = \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon_M \end{bmatrix} \qquad e_{ij} \,, \boldsymbol{e} \leftrightarrow \boldsymbol{\varepsilon}_D$$

deviatoric part / strain deviator

Training: Prove that the trace of $\boldsymbol{\varepsilon}_D$ is equal to 0.

Current summary:

6 unknowns σ_{ij} ⟶ equilibrium equation gives 3 relations

6 unknowns ε_{ij} ⟶ 6 kinematic equations relate strain tensor components to 3 displacement components

$$u_i\,,\quad i,j = 1,2,3 \qquad\qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Integrate the system of 6 equations to determine the 3 displacement components

$$\Downarrow$$

Compatibility equations (small strains) $\quad \varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0$

$$1.\quad \frac{\partial^2\varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$$

$$2.\quad \frac{\partial^2\varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2\varepsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}$$

$$3.\quad \frac{\partial^2\varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2\varepsilon_{11}}{\partial x_3^2} = 2\frac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_1}$$

$$4.\quad \frac{\partial}{\partial x_1}\left(-\frac{\partial\varepsilon_{23}}{\partial x_1} + \frac{\partial\varepsilon_{31}}{\partial x_2} + \frac{\partial\varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3}$$

$$5.\quad \frac{\partial}{\partial x_2}\left(\frac{\partial\varepsilon_{23}}{\partial x_1} - \frac{\partial\varepsilon_{31}}{\partial x_2} + \frac{\partial\varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}$$

$$6.\quad \frac{\partial}{\partial x_3}\left(\frac{\partial\varepsilon_{23}}{\partial x_1} + \frac{\partial\varepsilon_{31}}{\partial x_2} - \frac{\partial\varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}$$

Statically admissible set: $\{\sigma_{ij},\, F_i,\, T_i\}, \quad i,j = 1,2,3$

Any set of stresses σ_{ij} , body forces F_i and external forces T_i is a statically admissible set (equilibrium set) if it satisfies:

Conservation of energy

Kinematically admissible set: $\{\varepsilon_{ij},\, u_i\}, \quad i,j = 1,2,3$

Any set of displacements u_i and strains ε_{ij} is a kinematically admissible set (compatible set), if it satisfies:

Kinematic/
Geometric conditions

Statically admissible set: $\{\sigma_{ij},\, F_i,\, T_i\}, \quad i,j = 1,2,3$

Any set of stresses σ_{ij} , body forces F_i and external forces T_i is a statically admissible set (equilibrium set) if it satisfies:

Equation of equilibrium (motion) at interior points: $\sigma_{ij,j} + F_i = 0$ in Ω

Equilibrium of momentum at interior points: $\sigma_{ij} = \sigma_{ji}$ in Ω

Boundary condition at surface points where external forces act: $\sigma_{ij}\, n_j = T_i$ on $\partial\Omega_\sigma$

(SA1)

σ_{ij} is a statically admissible with $\{F_i, T_i\}$ stress field (stress state) if it satisfies (SA1) and this is NOT unique – in general, an infinity of stress states satisfies (SA1)

Conservation of energy

Kinematically admissible set: $\{\varepsilon_{ij},\, u_i\}\,, \quad i,j = 1,2,3$

Any set of displacements u_i and strains ε_{ij} is a kinematically admissible set (compatible set), if it satisfies:

Strain – displacements relation (infinitesimal deformation)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Compatibility conditions at interior points

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \qquad \Omega$$

Boundary condition at surface points

$$u_i = U_i \qquad \partial\Omega_u$$

(KA1)

ε_{ij} is a kinematically admissible with $\{u_i\}$ strain field if it satisfies (KA1) and this is NOT unique – in general, an infinite number of strain/displacements modes are compatible with a continuous distortion satisfying (KA1).

Kinematic/
Geometric conditions

Body forces; surface forces;
BC on $\partial\Omega_\sigma$

Displacements $\boldsymbol{u}$;
BC on $\partial\Omega_u$

equilibrium

compatibility

Static and kinematic conditions are both
independent of the material characteristics

Statically admissible stress field
σ_{ij}

Kinematically admissible strain field
ε_{ij}

not unique in
general case

Constitutive laws
material dependent relation

unique

Loading process **Deformation process**

is defined at the neighbourhood of the material point *P* of the body if the

stress / **strain** tensor

is given as a continouosly differentiable function of time, temperature and eventually other measurable physical characteristics.

Loading process ← dependence because of the material properties → **Deformation process**

⇓

At a given material point the loading and deformation processes can not be given independently. The dependence between these two processes is given by the constitutive law.

Real material behaviour is very complicated.
For mathematical convenience there are material properties idealization and subsequent classification of the materials laws.

Elastic body as idealization (material model simplification):

Definition (a): Elastic material (body) is called a material (body) for which at each material point the stress/ (or strain) is a unique function of strain (or stress):

$$\sigma_{ij} = \varphi_{ij}\left(\varepsilon_{kl}\right) \qquad \text{or} \qquad \varepsilon_{ij} = \varphi_{ij}^{-1}\left(\sigma_{kl}\right) \qquad \text{(E1)}$$

$\Rightarrow$ material behavior is time independent (there is only events consequence and no real time length) ;

$\Rightarrow$ path independence: strains are uniquely determined from the *current* state of stress and vice versa;

$\Rightarrow$ any process is reversible: to a closed stress path corresponds a closed strain path;

$\Rightarrow$ no dependence of the material behavior on the stress or strain history;

$\Rightarrow$ the process is isothermal (no influence of the temperature).

It is shown that Cauchy elastic material *may generate energy* under certain loading-unloading cycles and thus may violate the laws of (reversible) thermodynamics.

Cauchy elastic material

Eringen 1962

to Def. (a) and relations (E1) add the following restriction to the class of elastic materials:

(b): The work done over an elementary volume within a closed stress (or respectively strain) cycle is equal to zero.

- equivalent to the existence of stress (strain) potential -

$\Rightarrow$ (E1) becomes:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \quad \text{or} \quad \varepsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \qquad \text{(E2)}$$

with Legendre transform:

$$W\left(\varepsilon_{ij}\right) = \sigma_{kl}\varepsilon_{kl} - \Phi(\sigma_{ij})$$

Equations (E2) must be uniquely solved with respect to strain or stress $\Rightarrow$

$W\left(\varepsilon_{ij}\right) = const$ and $\Phi\left(\sigma_{ij}\right) = const$ must be not concave!

↑ elastic strain potential (strain energy density)

↑ elastic stress potential (complementary energy density)

Hyperelastic - Green elastic material

Green and Zerna 1954, Eringen 1962

Further simplification based on experiments (observations):

In case the deformations are small and the bady is elastic, then relations (E1) are *linear*.

$\Rightarrow$ For the Cauchy elastic material:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\,\varepsilon_{kl}$$

with B_{ij} -> initial stress tensor corresponding to the initial strain free state ($\varepsilon_{ij} = 0$).

C_{ijkl} -> tensor of material elastic constants (4th order tensor).

If the initial strain free state corresponds to the initial stress free space, $B_{ij} = 0$

$$\Downarrow$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

Generalized Hooke's law

3^4 = 81 constants for C_{ijkl} in general; σ_{ij} , ε_{ij} are symmetric -> max 36 are distinct

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

Further simplification based on experiments (observations):

In case the deformations are small and the bady is elastic, then relations (E1) are *linear*.

$\Rightarrow$ For the Green elastic material:

$$2\,W\left(\varepsilon_{ij}\right) = C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} \qquad (*)$$

Strain energy is a homogeneous quadratic function of the strain tensor components:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl}\,\varepsilon_{kl}$$

Generalized Hooke's law

From (*) it follows:

$$2\,W = C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} = C_{klij}\,\varepsilon_{ij}\varepsilon_{kl}$$

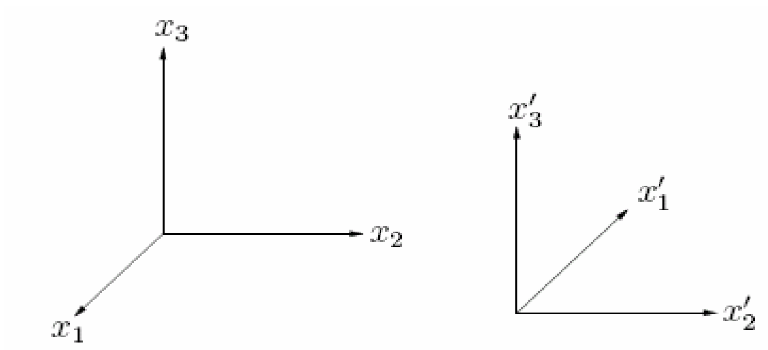
The number of maximum distinct components of C_{ijkl} reduces to 21.

Such an elastic material is called *linear elastic anisotropic material*.

Further simplification based on experiments (observations):

material symmetry properties:

most of the engineering materials possess some fabric (structure) symmetry and that means there are axes (planes) of symmetry that can be reversed without changing the material response.



One - symmetry plane

$$\boldsymbol{\alpha} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl}$$

$$\varepsilon'_{ij} = \alpha_{ik}\alpha_{jl}\varepsilon_{kl}$$

$$\sigma_{12} \rightarrow \sigma'_{12} = -\sigma_{12}$$

$$\sigma_{13} \rightarrow \sigma'_{13} = -\sigma_{13}$$

$$\varepsilon_{12} \rightarrow \varepsilon'_{12} = -\varepsilon_{12}$$

$$\varepsilon_{13} \rightarrow \varepsilon'_{13} = -\varepsilon_{13}$$

$$\left.\begin{aligned} \sigma_{ij} &= C_{ijkl}\,\varepsilon_{kl} \\ \sigma'_{ij} &= C_{ijkl}\,\varepsilon'_{kl} \end{aligned}\right\} \Rightarrow$$

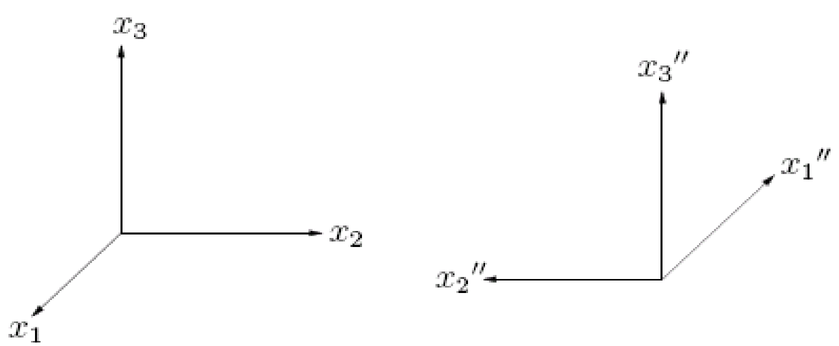
4+4=8 terms of C_{ijkl} are =0

21-8 = 13 elastic constants

Further simplification based on experiments (observations):

material symmetry properties:

orthotropic material



Two - symmetry plane

$$\boldsymbol{\alpha} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4 more terms of C_{ijkl} are = 0

$\Rightarrow$ **9 distinct constants**

$$\boldsymbol{\sigma} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \varepsilon$$

$$\boldsymbol{\sigma} = (\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{33} \;\; \sigma_{12} \;\; \sigma_{23} \;\; \sigma_{31})^T$$

$$\varepsilon = (\varepsilon_{11} \;\; \varepsilon_{22} \;\; \varepsilon_{33} \;\; 2\varepsilon_{12} \;\; 2\varepsilon_{23} \;\; 2\varepsilon_{31})^T$$

$$1 \mathrel{\hat{=}} 11,\; 2 \mathrel{\hat{=}} 22,\; 3 \mathrel{\hat{=}} 33,\; 4 \mathrel{\hat{=}} 12,\; 5 \mathrel{\hat{=}} 23,\; 6 \mathrel{\hat{=}} 31$$

Further simplification based on experiments (observations):

material symmetry properties:

transversely isotropic material

Special class of orthotropic materials that that have the same properties in one plane (e.g. the x_1-x_2 plane) and different properties in the direction normal to this plane (e.g. the x_3-axis) -> **5 distinct constants**

Further simplification based on experiments (observations):

material symmetry properties:

cubic symmetry

The properties along x_1-,x_2- and x_3- directions are identical -> **3 distinct constants**

$$C_{1111} = C_{2222} = C_{3333} \qquad C_{1222} = C_{2323} = C_{3131}$$

$$C_{1122} = C_{1133} = C_{2233}$$

Further simplification based on experiments (observations):

material symmetry properties:

isotropic material

For isotropic material, the elastic constants in:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

must be the same for ALL directions $\Rightarrow$

The tensor C_{ijkl} must be isotropic 4th order tensor

General form for an isotropic tensor of 4th order:

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\,\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + \alpha\,\left(\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right)$$

α =0 for symmetric tensors

symmetry

$$\Downarrow$$

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\,\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right)$$

Further simplification based on experiments (observations):

material symmetry properties:

isotropic material

$$\boldsymbol{\sigma} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & \text{sym.} & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

With Lame constants λ and μ :

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\delta_{ij}\,\varepsilon_{kk}$$

2 elastic constants

-> any pair of: shear modulus; Young's modulus; Poisson's ratio; bulk modulus

λ	μ	E	ν	K
[Pa]	[Pa]	[Pa]	[-]	[Pa]

Further simplification based on experiments (observations):

material symmetry properties:

isotropic material

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0 \\ & \textbf{sym.} & & & \frac{1}{2}(C_{11}-C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11}-C_{12}) \end{pmatrix}$$

This elastic matrix is representing an elastic constitutive tensor of order 4 that characterizes an isotropic material behaviour.

$$\boldsymbol{\sigma} = (\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{33} \;\; \sigma_{12} \;\; \sigma_{23} \;\; \sigma_{31})^T$$

$$\varepsilon = (\varepsilon_{11} \;\; \varepsilon_{22} \;\; \varepsilon_{33} \;\; 2\varepsilon_{12} \;\; 2\varepsilon_{23} \;\; 2\varepsilon_{31})^T$$

Elastodynamic problem for isotropic, homogeneous, linear elastic body:

Equilibrium

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \qquad \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

Hooke's law

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \qquad \boldsymbol{\sigma} = \lambda \mathbf{I} 3\varepsilon_M + 2\mu \mathbf{E}$$

Strain-displacement relation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \mathbf{E} = \frac{1}{2}(\mathbf{u}\nabla + \nabla \mathbf{u})$$

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i})$$

$,j$ or the same: $\dfrac{\partial}{\partial x_j}$

Displacement formulation – *Lame-Navier equations*:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} + f_i = \rho \ddot{u}_i \qquad \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}} \ .$$

3 equations; form with indices

form without indices

Next week – isotropic nonlinear elastic material law

Monday, 29th of October: seminar work

October 31 – Reformationstag (civic holiday in Thuringia)
1517, Martin Luther

October	November	Decembre
15.10.2007 – **L1** 17.10.2007 – **L2** + **HW1**	5.11.2007 – S2 -> **HW2** 7.11.2007 – **L5** +**HW3 +P**	3.12.2007 – S4 -> **HW4,5** 5.12.2007 – **L9**
22.10.2007 – **L3** 24.10.2007 – **L4** +**HW2**	12.11.2007 – **L6** +**HW4** 14.11.2007 – **L7**	10.12.2007 – **Projects presentations** 12.12. 2007 – **L10 + exam. questionnaire**
29.10.2007 – S1 -> **HW1** **31.10.2007 – ??**	19.11.2007 – **L8** +**HW5** **21.11.2007 – S3 -> HW3,4,5**	17.12. 2007 – S5 pre-exam **19.12.2007 EXAM**

P – projects: Si – seminar Number: Li – lecture Number; HWi – homework Number