Constitutive models: Strain Tensor

Principal strains, principal directions

\[ \varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \frac{1}{2} \gamma_{12} & \frac{1}{2} \gamma_{13} \\ \frac{1}{2} \gamma_{21} & \varepsilon_{22} & \frac{1}{2} \gamma_{23} \\ \frac{1}{2} \gamma_{31} & \frac{1}{2} \gamma_{32} & \varepsilon_{33} \end{pmatrix} \]

The principal strains are determined from the characteristic (eigenvalue) equation:

\[ |\varepsilon_{ij} - \varepsilon^{(k)} \delta_{ij}| = 0 \quad k = 1, 2, 3 \]

The three eigenvalues are the principal strains. The corresponding eigenvectors designate the direction (principal direction) associated with each of the principal strains:

\[ (\varepsilon_{ij} - \varepsilon^{(k)} \delta_{ij}) n_i^{(k)} = 0 \]

! In general the principal directions for the stress and the strain tensors do not coincide.
Constitutive models: Strain Tensor

Strain invariants:

\[ |\varepsilon_{ij} - \varepsilon^{(k)} \delta_{ij}| = \]

\[ (\varepsilon^{(k)})^3 - I_1^{\varepsilon} (\varepsilon^{(k)})^2 + I_2^{\varepsilon} (\varepsilon^{(k)}) - I_3^{\varepsilon} = 0 \]

\[ I_1^{\varepsilon} = \varepsilon_{ii} = tr\varepsilon \]

\[ I_2^{\varepsilon} = \frac{1}{2} (\varepsilon_{ii} \varepsilon_{jj} - \varepsilon_{ij} \varepsilon_{ij}) \]

\[ I_3^{\varepsilon} = \text{det } \varepsilon \]

Note: \[ \nabla \cdot \mathbf{u} = u_{i,i} = \varepsilon_{ii} \]
Constitutive models: Strain Tensor

Decomposition to spherical (hydrostatic) and deviatoric parts

\[ \varepsilon_{ij} = e_{ij} + \varepsilon_M \delta_{ij} \]

\[ \varepsilon_M = \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix} \]

\[ \varepsilon_M = \frac{\varepsilon_{ii}}{3} \]

spherical or hydrostatic part

\[ \varepsilon_D = \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon_M \end{bmatrix} \]

deviatoric part / strain deviator

Training: Prove that the trace of \( \varepsilon_D \) is equal to 0.
Constitutive models: Stress — Strain Relations

Current summary:

6 unknowns $\sigma_{ij}$  $\rightarrow$ equilibrium equation gives 3 relations

6 unknowns $\varepsilon_{ij}$  $\rightarrow$ 6 kinematic equations relate strain tensor components to 3 displacement components

$u_i, i, j = 1, 2, 3$

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

Integrate the system of 6 equations to determine the 3 displacement components

Compatibility equations (small strains)

1. $\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$

2. $\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2 \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3}$

3. $\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = 2 \frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1}$

4. $\frac{\partial}{\partial x_1} \left( -\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3}$

5. $\frac{\partial}{\partial x_2} \left( \frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1}$

6. $\frac{\partial}{\partial x_3} \left( \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}$
Constitutive models: Stress – Strain Relations

Statically admissible set: \( \{ \sigma_{ij}, F_i, T_i \} \), \( i, j = 1, 2, 3 \)

Any set of stresses \( \sigma_{ij} \), body forces \( F_i \) and external forces \( T_i \) is a 
statically admissible set (equilibrium set) if it satisfies:

Conservation of energy

Kinematically admissible set: \( \{ \varepsilon_{ij}, u_i \} \), \( i, j = 1, 2, 3 \)

Any set of displacements \( u_i \) and strains \( \varepsilon_{ij} \) is a kinematically 
admissible set (compatible set), if it satisfies:

Kinematic/
Geometric conditions
Constitutive models: Stress — Strain Relations

**Statically admissible set:** \( \{\sigma_{ij}, F_i, T_i\} \), \( i, j = 1, 2, 3 \)

Any set of stresses \( \sigma_{ij} \), body forces \( F_i \) and external forces \( T_i \) is a statically admissible set (equilibrium set) if it satisfies:

\[
\begin{align*}
\text{Equation of equilibrium (motion)} & : & \sigma_{ij,j} + F_i &= 0 & (\Omega) \\
\text{Equilibrium of momentum} & : & \sigma_{ij} &= \sigma_{ji} \\
\text{Boundary condition} & : & \sigma_{ij} n_j &= T_i & (\partial \Omega_\sigma)
\end{align*}
\]

\( \sigma_{ij} \) is a statically admissible with \( \{F_i, T_i\} \) stress field (stress state) if it satisfies (SA1) and this is NOT unique – in general, an infinity of stress states satisfies (SA1)

Conservation of energy
Constitutive models: Stress — Strain Relations

**Kinematically admissible set:** \( \{ \varepsilon_{ij}, u_i \} \), \( i, j = 1, 2, 3 \)

Any set of displacements \( u_i \) and strains \( \varepsilon_{ij} \) is a kinematically admissible set (compatible set), if it satisfies:

**Strain – displacements relation**
( infinitesimal deformation)

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right)
\]

**Compatibility conditions**
at interior points

\[
\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0
\}

\( \Omega \)

(KA1)

**Boundary condition**
at surface points

\[
u_i = U_i \quad \partial\Omega_u
\]

\( \varepsilon_{ij} \) is a kinematically admissible with \( \{ u_i \} \) strain field if it satisfies (KA1) and this is NOT unique – in general, an infinite number of strain/displacements modes are compatible with a continuous distortion satisfying (KA1).
Constitutive models: Stress — Strain Relations

Body forces; surface forces; BC on $\partial \Omega_\sigma$

Displacements $u$; BC on $\partial \Omega_u$

Statically admissible stress field $\sigma_{ij}$

Kinematically admissible strain field $\varepsilon_{ij}$

Static and kinematic conditions are both independent of the material characteristics

Constitutive laws
material dependent relation

not unique in general case

unique
Constitutive models: Stress – Strain Relations

**Loading process**

is defined at the neighbourhood of the material point $P$ of the body if the

**stress / strain** tensor

is given as a continuously differentiable function of time, temperature and eventually other measurable physical characteristics.

At a given material point the loading and deformation processes cannot be given independently. The dependence between these two processes is given by the constitutive law.

Real material behaviour is very complicated. For mathematical convenience there are material properties idealization and subsequent classification of the materials laws.
Elastic body as idealization (material model simplification):

Definition (a): Elastic material (body) is called a material (body) for which at each material point the stress/ (or strain) is a unique function of strain (or stress):

\[ \sigma_{ij} = \varphi_{ij}(\varepsilon_{kl}) \quad \text{or} \quad \varepsilon_{ij} = \varphi_{ij}^{-1}(\sigma_{kl}) \]  \hspace{1cm} (E1)

⇒ material behavior is time independent (there is only events consequence and no real time length);

⇒ path independence: strains are uniquely determined from the current state of stress and vice versa;

⇒ any process is reversible: to a closed stress path corresponds a closed strain path;

⇒ no dependence of the material behavior on the stress or strain history;

⇒ the process is isothermal (no influence of the temperature).

It is shown that Cauchy elastic material may generate energy under certain loading-unloading cycles and thus may violate the laws of (reversible) thermodynamics.

**Cauchy elastic material**
Constitutive models: Elastic Stress – Strain Relations

To Def. (a) and relations (E1) add the following restriction to the class of elastic materials:

(b): The work done over an elementary volume within a closed stress (or respectively strain) cycle is equal to zero.

- equivalent to the existence of stress (strain) potential -

\[ \Rightarrow \quad (E1) \text{ becomes:} \]

\[
\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \quad \text{or} \quad \varepsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \]

(E2)

with Legendre transform:

\[ W(\varepsilon_{ij}) = \sigma_{kl} \varepsilon_{kl} - \Phi(\sigma_{ij}) \]

Equations (E2) must be uniquely solved with respect to strain or stress \( \Rightarrow \)

\[ W(\varepsilon_{ij}) = \text{const} \quad \text{and} \quad \Phi(\sigma_{ij}) = \text{const} \]

must be not concave!

[\text{elastic strain potential (strain energy density)}]

[\text{elastic stress potential (complementary energy density)}]

Hyperelastic - Green elastic material
Constitutive models: Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):

In case the deformations are small and the body is elastic, then
relations (E1) are linear.

 ⇒ For the Cauchy elastic material:

\[ \sigma_{ij} = B_{ij} + C_{ijkl} \varepsilon_{kl} \]

with \( B_{ij} \) -> initial stress tensor corresponding to the initial strain free state (\( \varepsilon_{ij} = 0 \)).

\( C_{ijkl} \) -> tensor of material elastic constants (4th order tensor).

If the initial strain free state corresponds to the initial stress free space, \( B_{ij} = 0 \)

\[ \downarrow \]

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]

Generalized Hooke’s law

\( 3^4 = 81 \) constants for \( C_{ijkl} \) in general; \( \sigma_{ij}, \varepsilon_{ij} \) are symmetric -> max 36 are distinct

\[ C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} \]
Constitutive models: Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):
In case the deformations are small and the body is elastic, then relations (E1) are linear.

⇒ For the Green elastic material:

\[ 2W \left( \varepsilon_{ij} \right) = C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \]  \hspace{1cm} (*)

Strain energy is a homogeneous quadratic function of the strain tensor components:

\[ \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} \] \hspace{1cm} Generalized Hooke's law

From (*) it follows:

\[ 2W = C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = C'_{klij} \varepsilon_{ij} \varepsilon_{kl} \]

The number of maximum distinct components of \( C_{ijkl} \) reduces to 21.

Such an elastic material is called linear elastic anisotropic material.
Constitutive models: Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):

**Material symmetry properties:**
most of the engineering materials possess some fabric (structure) symmetry and that means there are axes (planes) of symmetry that can be reversed without changing the material response.

\[
\alpha = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl}
\]

\[
\varepsilon'_{ij} = \alpha_{ik}\alpha_{jl}\varepsilon_{kl}
\]

\[
\begin{align*}
\sigma_{12} \rightarrow \sigma'_{12} &= -\sigma_{12} \\
\sigma_{13} \rightarrow \sigma'_{13} &= -\sigma_{13} \\
\varepsilon_{12} \rightarrow \varepsilon'_{12} &= -\varepsilon_{12} \\
\varepsilon_{13} \rightarrow \varepsilon'_{13} &= -\varepsilon_{13}
\end{align*}
\]

\[
\begin{align*}
\sigma'_{ij} &= C'_{ijkl}\varepsilon_{kl} \\
\sigma_{ij} &= C_{ijkl}\varepsilon_{kl}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow 4+4=8 \text{ terms of } C'_{ijkl} \text{ are } =0 \\
21-8 = 13 \text{ elastic constants}
\end{align*}
\]
Constitutive models: Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):

**Material symmetry properties:**

**Orthotropic material**

\[ \mathbf{\alpha} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Two-symmetry plane

4 more terms of \( C_{ijkl} \) are \( = 0 \)

\[ \Rightarrow \] 9 distinct constants

\[ \begin{align*}
\mathbf{\sigma} &= \begin{pmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} \end{pmatrix}^T \\
\mathbf{\varepsilon} &= \begin{pmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{12} & 2\varepsilon_{23} & 2\varepsilon_{31} \end{pmatrix}^T
\end{align*} \]

\( 1 \approx 11, 2 \approx 22, 3 \approx 33, 4 \approx 12, 5 \approx 23, 6 \approx 31 \)
Constitutive models: Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):
material symmetry properties:
transversely isotropic material

Special class of orthotropic materials that that have the same properties in one plane (e.g. the $x_1$-$x_2$ plane) and different properties in the direction normal to this plane (e.g. the $x_3$-axis) -> 5 distinct constants

Further simplification based on experiments (observations):
material symmetry properties:
cubic symmetry

The properties along $x_1$-$, x_2$- and $x_3$- directions are identical -> 3 distinct constants

\[ C_{1111} = C_{2222} = C_{3333} \quad C_{1222} = C_{2323} = C_{3131} \]
\[ C_{1122} = C_{1133} = C_{2233} \]
Constitutive models: Isotropic Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):

material symmetry properties:
isotropic material

For isotropic material, the elastic constants in:

\[ \sigma_{ij} = C'_{ijkl} \varepsilon_{kl} \]

must be the same for ALL directions \[ \Rightarrow \]

The tensor \[ C'_{ijkl} \] must be isotropic 4th order tensor

General form for an isotropic tensor of 4th order:

\[
C'_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \alpha (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})
\]

symmetry

\[ \Rightarrow \]

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

=0 for symmetric tensors
Constitutive models: Isotropic Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):
material symmetry properties:
isotropic material

\[ \sigma = \begin{bmatrix}
2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\
2\mu + \lambda & \lambda & 0 & 0 & 0 & 0 \\
2\mu + \lambda & 0 & 0 & 0 & 0 & 0 \\
\text{sym.} & 2\mu & 0 & 0 & 0 & 0 \\
2\mu & 0 & 0 & 0 & 0 & 0 \\
2\mu & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{23} \\
\varepsilon_{31} \\
\end{bmatrix} \]

With Lame constants \( \lambda \) and \( \mu \):

\[ \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk} \]

2 elastic constants
-> any pair of: shear modulus; Young’s modulus; Poisson’s ratio; bulk modulus

| \( \lambda \) | \( \mu \) | \( E \) | \( \nu \) | \( K \) |
| Pa | Pa | Pa | - | Pa |

24.10.2007
Constitutive models: Isotropic Linear Elastic Material – Generalized Hooke’s Law

Further simplification based on experiments (observations):

material symmetry properties:

isotropic material

\[
\begin{bmatrix}
    C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
    C_{11} & C_{12} & 0 & 0 & 0 & 0 \\
    C_{11} & 0 & 0 & 0 & 0 & 0 \\
    \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 & 0 & 0 & 0 \\
    \text{sym.} & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 & 0 & 0 \\
    \frac{1}{2}(C_{11} - C_{12}) & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

This elastic matrix is representing an elastic constitutive tensor of order 4 that characterizes an isotropic material behaviour.

\[
\sigma = (\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12} \sigma_{23} \sigma_{31})^T
\]

\[
\varepsilon = (\varepsilon_{11} \varepsilon_{22} \varepsilon_{33} 2\varepsilon_{12} 2\varepsilon_{23} 2\varepsilon_{31})^T
\]

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Constitutive models: Isotropic Linear Elastic Material – Generalized Hooke’s Law

Elastodynamic problem for isotropic, homogeneous, linear elastic body:

**Equilibrium**

\[ \sigma_{ij,j} + f_i = \rho \ddot{u}_i \quad \nabla \cdot \sigma + f = \rho \ddot{u} \]

**Hooke’s law**

\[ \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 \mu \varepsilon_{ij} \quad \sigma = \lambda I^3 \varepsilon_M + 2 \mu E \]

**Strain-displacement relation**

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad E = \frac{1}{2} (u \nabla + \nabla u) \]

**Displacement formulation – Lame-Navier equations:**

\[ \mu u_{ij,j} + (\lambda + \mu) u_{i,j,j} + f_i = \rho \ddot{u}_i \quad \mu \nabla^2 u + (\lambda + \mu) \nabla \cdot u + f = \rho \ddot{u} . \]

3 equations; form with indices

form without indices
Constitutive models: Isotropic Nonlinear Elastic Material

Next week – isotropic nonlinear elastic material law

Monday, 29th of October: seminar work

October 31 – Reformationstag (civic holiday in Thuringia)
1517, Martin Luther
### Constitutive models: Isotropic Nonlinear Elastic Material

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**P** – projects; **Si** – seminar; **Number:** **Li** – lecture Number; **HWi** – homework Number