Constitutive models - Introduction

✓ Traction, stress and equilibrium

✓ Stress tensor

✓ Tensor’s invariants (2nd order)
  ✓ Stress tensor invariants

➤ Deformation / strain tensor

http://www.uni-weimar.de/cms/Constitutive-models.5613.0.html?&L=%21
Constitutive models – strain tensor → coming

Next:

Position vector – displacement vector in Lagrangian and Eulerian description (very brief)

Deformation

Strain tensor

Some geometrical representations

**Constitutive model 1**: Generalized Hooke’s law
Constitutive models – Strain Analysis

Position vector – displacement vector
Lagrangian and Eulerian description

initial/undeformed configuration

$P(x)$

$t = 0$

$X$

current/deformed configuration

$p(X)$

$t = t$

Strain analysis concerns a geometrical problem since it studies the geometrical expression of the body deformation due to applied forces.

This analysis is not related to the material properties and it is NOT leading to constitutive relations.
Rigid body motion

If the distance between every pair of point (particles) in the body remains constant during the body motion, then the body is said to experience rigid body motion.

\[ u = X - x \]
\[ u_i = X_i - x_i. \]
Constitutive models – Strain Analysis

The movement of the point (particle) \( P \) is called to be known (defined) if at each time \( t \) the relation between \( X \) and \( x \) is a known vector-function:

\[
X = X(x, t) = \phi(x, t)
\]

\( e_1, e_2, e_3 \rightarrow \text{defined (fixed) normalized 3D orthogonal basis} \)

* these are 3 scalar relations
Constitutive models – Strain Analysis

\[ X = X(x, t) = \phi(x, t) \]  \hspace{1cm} (L1)

**Lagrangian** description; Lagrange (material) coordinates - curvilinear, nonorthogonal coordinates (only at \( t=t_0 \) the Lagrangian coordinates are Cartesian)

The mapping (L1) is unique and invertible if its Jacobian is such that:

\[ J = \left| \frac{\partial X_i}{\partial x_j} \right| = |X_{i,j}| \neq 0 \]

Inversion of (L1) gives:

\[ x = x(X, t) \]  \hspace{1cm} (E1)

**Eulerian** description; Euler (spatial) coordinates – fixed Cartesian coordinate system

**Displacement and velocity vectors (definition):**

\[ u = X - x \]  \hspace{1cm} + (L1) \hspace{1cm} v = \frac{\partial X}{\partial t} \]  \hspace{1cm} + (E1)

used by Lagrange method as basic kinematical characteristic

used by Euler method as basic kinematical characteristic
Constitutive models – Strain Analysis

Strain tensor

initial/undeformed configuration current/deformed configuration

expression of the distance by the Jacobian of the mapping

\[ |ds|^2 = dx_i dx_i \]

\[ |dS|^2 = dX_i dX_i \]

\[ |dS|^2 = dx_i \frac{\partial x_i}{\partial X_j} \frac{\partial x_i}{\partial X_k} dX_j dX_k \]

\[ |dS|^2 = dX_i \frac{\partial X_i}{\partial x_j} \frac{\partial X_i}{\partial x_k} dx_j dx_k. \]

fibre PQ -> fibre pq

\[ ds + u + du - dS - u = 0 \]
\[ \Rightarrow du = dS - ds. \]

* deformation gradient

\[ F_{ij} = X_{i,j} \]
Constitutive models – Strain Analysis

Strain tensor

The distance in material (Lagrangian) coordinates:

\[ ds + u + du - dS - u = 0 \]

\[ \Rightarrow du = dS - ds. \]

The distance in material (Lagrangian) coordinates:

\[
|ds|^2 = X_{i,j}X_{i,k}dx_jdx_k - dx_i dx_i
\]

\[
= (X_{i,j}X_{i,k} - \delta_{jk}) dx_j dx_k - 2\varepsilon_{j,k}^L
\]

Green or Lagrangian strain tensor

fibre PQ -> fibre pq

initial/undeformed configuration
current/deformed configuration
Constitutive models – Strain Analysis

Strain tensor

The distance in spatial (Eulerian) coordinates:

\[ |dS|^2 - |ds|^2 = |dX_i dX_i - \partial X_i \partial X_i| dX_j dX_k \]

\[ = (\delta_{jk} - \epsilon_{i,j} \epsilon_{i,k}) dX_j dX_k \]

\[ = 2\epsilon_{j,k} \] (E42)

Euler or Almansi strain tensor
Constitutive models – Strain Analysis

Strain tensor
represents the deformation of a point (particle) vicinity

Green or Lagrangian strain tensor

\[ \varepsilon_{jk}^L = \frac{1}{2} \left[ (\delta_{ij} - \delta_{ij}) (u_{i,k} + \delta_{ik}) - \delta_{jk} \right] \]

\[ \Rightarrow \left\{ \begin{array}{l}
\varepsilon_{jk}^L = \frac{1}{2} \left[ u_{k,j} + u_{j,k} + \delta_{ij} u_{i,k} + \delta_{jk} - \delta_{jk} \right] \\
\varepsilon_{ik}^L = \frac{1}{2} \left[ u_{k,i} + u_{j,k} + \delta_{ij} u_{i,k} \right]
\end{array} \right. \] (L41)

Euler or Almansi strain tensor

\[ \varepsilon_{jk}^E = \frac{1}{2} \left[ \delta_{jk} - (\delta_{ij} - \delta_{ij}) (\delta_{ik} - u_{i,k}) \right] \]

\[ \Rightarrow \left\{ \begin{array}{l}
\varepsilon_{jk}^E = \frac{1}{2} \left[ u_{k,j} + u_{j,k} - \delta_{ij} u_{i,k} \right]
\end{array} \right. \] (E42)

For small displacement gradient (infinitesimal (small) deformations) Lagrangian and Euler strain tensors coincide (also the methods of description of motion):

\[ \varepsilon_{ij} = \varepsilon_{ij}^L = \varepsilon_{ij}^E = \frac{1}{2} (u_{i,j} + u_{j,i}) \]

* symmetric 2nd order tensor
Constitutive models – Strain Analysis – Small Strains

Current summary:

6 unknowns $\sigma_{i,j}$ $\rightarrow$ equilibrium equation gives 3 relations

6 unknowns $\varepsilon_{i,j}$ $\rightarrow$ 6 kinematic equations relate strain tensor components to 3 displacement components

$$u_i, \ i, j = 1, 2, 3 \quad \varepsilon_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Integrate the system of 6 equations to determine the 3 displacement components

\[ \downarrow \]

 Compatibility equations (small strains)

\begin{align*}
1. & \quad \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{21}}{\partial x_1 \partial x_2} \\
2. & \quad \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2 \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} \\
3. & \quad \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = 2 \frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} \\
4. & \quad \frac{\partial}{\partial x_1} \left( - \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} \\
5. & \quad \frac{\partial}{\partial x_2} \left( \frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} \\
6. & \quad \frac{\partial}{\partial x_3} \left( \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}
\end{align*}