Constitutive models – Introduction – engineering application

Pilot test (name, e-mail, main topic of interest):

1. 
\[ f(x_1, x_2, x_3) = 3x_1 + x_1 e^{x_2} + x_1 x_2 e^{x_3} \]

(a) grad \( f \) = ?
(b) grad \( f(3, 1, 0) \) = ?

2. Is the equation of motion a constitutive equation?

3. Give the name of at least one constitutive law you know.

3. For the tensor \( T_{ij} \), it holds
\[ T_{ij} = -T_{ji} \quad i, j = 1, 2, 3 \] .
Write the tensor \( T_{ij} \) as a matrix.
Give the rank (order) of the tensor \( T_{ij} \).

4. In 2D a coordinate transform \( x_i \rightarrow x'_i \) is defined with the angle between \( x_1 \) and \( x'_1 \) equal to \( \pi / 2 \). Give the rotational matrix \( \alpha_{ij} \).
Constitutive models – Introduction – engineering application

Engineering Problem

Boundary and initial value problem
geometry, applied constrains, forces etc.

Starting point for deriving partial differential equations (PDE)
lies in the conservative laws of physics

Conservation of mass
(Continuity equation)

Conservation of energy
(Equation of equilibrium/motion)

general laws

Add constitutive relationships
(stress-strain relationships)

SOLVE the PDE – BVP
numerically: FE; FD; FV etc

PREDICTIONS

nonlinear PDE,
geometrical nonlinearity:
analytical solution rarely possible
Mass is a fundamental concept in physics, roughly corresponding to the intuitive idea of "how much matter there is in an object".

In physics, acceleration is defined as the rate of change of velocity,

Velocity is defined as the rate of change of position.

Position - location in a coordinate system
Constitutive models – Introduction – Pilot test discussion

Equation of motion and does it present a constitutive law?
NO – it is a general law and it is material independent!
Everything „moves“ or is in a „static“ equilibrium.

Constitutive <=> material law

A constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material or substance, and does not follow directly from physical law. It is combined with other equations that do represent physical laws to solve some physical problem.

Voting recapitulation on 15th of October 2007: YES 3; NO 4; X 3

Voting recapitulation on 19th of December 2007: coming soon
Constitutive models – Introduction – Pilot test discussion

i.e., conceptual and mathematical mechanics
Truesdell, Noll, etc

The modern mechanical theories regard bodies as subject to general laws applying to all types of materials, laws that characterize geometry of space, time and motion, the structure of material bodies, the nature of systems of forces and the relation of force to motion.

Mechanics uses the term constitutive assumption to refer to the special laws for particular materials, since these laws reflect assumptions about the constitution of the material.

Mechanical practice depends critically on these special laws.

Constitutive assumptions – relations – laws – equations are the TOPIC of this lecture course
Constitutive models – Introduction – Pilot test discussion

Michopoulos, Farhat, Fish, 2005

equation of motion

Hooke’s law

Michopoulos, Farhat, Fish, 2005
Constitutive models - Introduction

Traction, stress and equilibrium

Stress tensor

Deformation / strain tensor

Tensor’s invariants (2nd order)
Constitutive models – Traction, stress, equilibrium

Traction and couple-stress vector

deformable body under load (external + internal)

\[ \lim_{\Delta A_n \to 0} \frac{\Delta F_n}{\Delta A_n} = \frac{dF_n}{dA_n} = t_n \]

traction vector
limiting force intensity

\[ \lim_{\Delta A_n \to 0} \frac{\Delta M_n}{\Delta A_n} = \frac{dM_n}{dA_n} = C_n \]

couple-stress vector

=0 if no particle rotation is considered
else the continua are called Cosserat-Continua and it is not included in this lecture course

\[ \Delta A_n \text{ is small and } \Rightarrow \Delta F_n \text{ is almost constant over } \Delta A_n \]
Constitutive models – Traction, stress, equilibrium

Stress tensor

\[ t_i = \sigma_{i1} e_1 + \sigma_{i2} e_2 + \sigma_{i3} e_3 \]

\[ t_i = \sigma_{ij} e_j. \]

\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \sigma_{ij}, \]
Constitutive models – Traction, stress, equilibrium

Stress tensor

\[ \frac{1}{3} dA_\beta \, dx_\beta = \frac{1}{3} dA_n \, h, \quad \beta = 1, 2, 3 \]

\[ \text{no summation over } \beta \]

\[ dA_\beta = dA_n \, e_\beta \cdot n = dA_n \, n_\beta \]

Equilibrium of forces at the tetrahedron:

\[ t_n \, dA_n - t_i \, dA_i + f \frac{1}{3} h \, dA_n = 0 \quad \Rightarrow \]

\[ t_n - t_i n_i + f \frac{1}{3} h = 0, \quad h \to 0 \quad \Rightarrow \quad t_n = t_i n_i = \sigma_{ij} e_j n_i \]

\[ t_i = \sigma_{ij} n_j \quad \Leftarrow \quad t_n = t_i e_i \]

\[ \text{summation over } i \]
Constitutive models – Traction, stress, equilibrium

At a given body point \( P \) the traction in a given direction \( n \) is uniquely determined by the stress tensor \( \sigma_{ij} \)

Stress tensor is a 2nd order tensor => it is represented by 9 numbers:

\[
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix} = \sigma_{ij}
\]
Equilibrium is a general property concerned with the conservation of energy => laws related to it are general laws, material independent and NOT constitutive laws.

1. Linear momentum principle: consider arbitrary body with volume $V$ and surface boundary $A$ =>

$$\int_V \mathbf{f} \, dV + \int_A \mathbf{t} \, dA = \int_V \rho \frac{d^2 \mathbf{u}}{dt^2} \, dV$$

with displacement vector $\mathbf{u}$ and density $\rho$.

\[\downarrow\]

Equation of motion:

$$\nabla \cdot \mathbf{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

As $i=1,2,3$ => these are 3 equations for the unknown stress
Constitutive models – Traction, stress, equilibrium

Equilibrium is a general property concerned with the conservation of energy => laws related to it are general laws, material independent and NOT constitutive laws.

2. Momentum equilibrium principle: consider arbitrary body with volume $V$ and surface boundary $A$ =>

$$\int_{V} (r \times f) \, dV + \int_{A} (r \times t) \, dA = \int_{V} (r \times \rho \ddot{u}) \, dV$$

with displacement vector $\mathbf{u}$ and density $\rho$.

$\Downarrow$

Gives that stress tensor is a symmetric 2nd order tensor:

$$\sigma_{ij} = \sigma_{ji} \quad \sigma = \sigma^T \quad \Rightarrow \quad 6 \text{ numbers define the stress tensor at a given body point.}$$
1. Linear momentum principle

\[ \nabla \cdot \mathbf{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}} \]
\[ \sigma_{ji,j} + f_i = \rho \ddot{u}_i \]

As \( i=1,2,3 \Rightarrow \)
these are 3 equations for the unknown stress tensor components.

2. Momentum equilibrium principle

\[ \sigma_{ij} = \sigma_{ji} \]
\[ \sigma = \sigma^T \quad \Rightarrow \]

6 numbers define the stress tensor at a given body point.

↓

There are 3 equations missing to complete the system of PDE for the unknown stress tensor. These missing equations will come due to the **constitutive relations**.
Constitutive models – stress tensor –> back to the tensor algebra

Principal axes, principal stresses

Question: Is there in any body at any particular body point a plane where the area element experiences only normal stresses?

Answer: It is known from the linear algebra that there exists an orthogonal transformation which transforms a symmetric matrix to a diagonal one => yes and for determining these planes the eigenvalue problem has to be solved, namely to solve:

\[ |\sigma_{ij} - \sigma^{(k)} \delta_{ij}| = 0 \quad \iff \quad \sigma = \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{pmatrix} \]

principal values

\( \sigma^{(k)} \) - (eigenvalues) principal stresses

\[ (\sigma_{ij} - \sigma^{(k)} \delta_{ij}) n_j = 0 \quad \iff \quad n_i^{(k)} n_i^{(k)} = 1 \]

principal axes

to determine (eigenvectors) principal directions

17.10.2007
Principal axes, principal stresses

1. All principal directions are orthogonal if the three principal stresses are distinct => there are 3 numbers (principal stresses) and 3 angles (directions) for determining the stress in the principal (main) coordinate system.

\[ \sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)} \]
2. All the three principal stresses and the three principal directions are REAL. (this can be easily proved using the symmetry property of the stress tensor and the fact that the eigenvalue equation is a polynomial of 3rd order).

Summary: If the three roots of the eigenvalue problem are different => there exist three mutually orthogonal directions at point \( P \) such that area elements perpendicular to these directions experience only NORMAL stresses and these directions are called **principal directions at point** \( P \). The corresponding normal stresses are called **principal stresses**.
Stress invariants, deviatoric and spherical decomposition

In general, the stress tensor at a distinct point differs in its form for different coordinate systems.

Previous knowledge: Eigenvalues are invariant regarding orthogonal coordinate transformations (rotations)

1. Principal stresses are invariants of the stress tensor.

2. As the eigenvalue equation is form-invariant regarding orthogonal transformation of the coordinate system => its coefficients are also being stress invariants:

\[ |\sigma_{ij} - \sigma^{(k)} \delta_{ij}| = (\sigma^{(k)})^3 - I_1 (\sigma^{(k)})^2 + I_2 \sigma^{(k)} - I_3 = 0 \]

\[ I_1 = \sigma_{ii} = tr \sigma \]

\[ I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ij}) \]

\[ I_3 = |\sigma_{ij}| = \det \sigma \]

\[ I_1 = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} \]

\[ I_2 = (\sigma^{(1)} \sigma^{(2)} + \sigma^{(2)} \sigma^{(3)} + \sigma^{(3)} \sigma^{(1)}) \]

\[ I_3 = \sigma^{(1)} \sigma^{(2)} \sigma^{(3)} \]

(coefficients depend on the stress tensor components)
Constitutive models – stress tensor --> back to the tensor algebra

\[ I_1 = \sigma_{ii} = tr \sigma \]
\[ I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ij}) \]
\[ I_3 = |\sigma_{ij}| = \det \sigma \]

\( I_1, I_2, \) and \( I_3 \) are **stress invariants** as they do not change in value when the axes are rotated to new positions.

(Previous knowledge: **Invariants** are scalar functions of tensors that by definition have the same value no matter the coordinate system to which they are referenced. )

! It is important to understand **stress invariants** since they play important role in developing constitutive equations.

!
Constitutive models – stress tensor \(\rightarrow\) back to the tensor algebra

Deviatoric and spherical decomposition

\[
\sigma = s + \sigma_m I = s + \frac{1}{3} I_1 I
\]

\[
\sigma_{ij} = s_{ij} + \sigma_m \delta_{ij} = s_{ij} + \frac{1}{3} I_1 \delta_{ij} = s_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij}
\]

spherical part of the stress tensor \(\sigma\)

\[
s = \begin{pmatrix}
\sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \sigma_m & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_m
\end{pmatrix}
\]

deviatoric part of the stress tensor \(\sigma\)
Constitutive models – stress tensor → back to the tensor algebra

Other in common use forms for the three stress invariants:

\[
\begin{align*}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \quad \text{first stress invariant (polynomial of degree 1)} \\
I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \quad \text{second stress invariant (polynomial of degree 2)} \\
I_3 &= \sigma_1\sigma_2\sigma_3 \quad \text{third stress invariant (polynomial of degree 3)}
\end{align*}
\]

\[
\begin{align*}
I_\sigma &= \text{tr } \sigma = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 = I_1 \\
II_\sigma &= \text{tr } \sigma^2 = \sigma_{ij}\sigma_{ij} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = I_1^2 - 2 I_2 \\
III_\sigma &= \text{tr } \sigma^3 = \sigma_{ik}\sigma_{kj}\sigma_{ji} = \sigma_1^3 + \sigma_2^3 + \sigma_3^3 = I_1^3 - 3 I_1 I_2 + 3 I_3
\end{align*}
\]

stress deviator invariants (1st is equal to 0)

\[
\begin{align*}
II_s &= \frac{1}{2} s_{ij}s_{ij} = \frac{1}{2} \left( II_\sigma - \frac{1}{3} I_\sigma^2 \right) \\
III_s &= \frac{1}{3} s_{ij}s_{jk}s_{ki} = \frac{1}{3} \left( III_\sigma - I_\sigma II_\sigma - \frac{2}{9} I_\sigma^3 \right)
\end{align*}
\]

Elementary: Any combination of any stress invariants is a stress invariant.

Not so elementary: the polynomial representation for the stress invariants is of max 3rd degree.
Any stress invariant can be represented using \( I_1, I_2, I_3 \).

* Notation for the principal stresses that is used more often is with subscript index:
\[
\sigma^{(i)} \leftrightarrow \sigma_i
\]
Next:

Position vector – displacement vector in Lagrangian and Eulerian description (very brief)

Deformation

Strain tensor

Some geometrical representations

**Constitutive model 1**: Generalized Hooke's law