NUMERICAL ANALYSIS OF STRONG DISCONTINUITY PROPAGATION IN DILATANT MEDIA: ENHANCED STRAIN FINITE ELEMENTS AND TRACKING STRATEGY

WENQING WANG, OLAF KOLDITZ

Environmental Informatics, Helmholtz Centre for Environmental Research – UFZ, Leipzig, D-04318, Germany,
e-mails: wenqing.wang@ufz.de, olaf.kolditz@ufz.de

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ABSTRACT. In this work a new approach is presented for discontinuity propagation in dilatant media. This model is a combination of enhanced strain finite elements and a specific tracking algorithm for discontinuity propagation. A non-associative flow rule is used until onset of discontinuity for plasticity calculation. For the evolution of the discontinuity (i.e. post localization) an associative flow rule is applied to derive a failure model. The material behaviour is represented by Drucker-Prager plasticity. Quadratic smooth functions are utilized in order to improve the kinematic properties of elements having discontinuities. A new discontinuity tracking technique is developed for 2D problems. This algorithm is efficient because tracking takes place only on neighbouring elements of the last ones where discontinuities have occurred. Three applications are presented which cover typical problems from geomechanics, such as biaxial and shear tests as well as footing problems.

KEY WORDS: strong discontinuity, discontinuity propagation, tracking, enhanced strain finite elements.

1. Introduction

Strain localization in geological media, such as rock or soil, can occur under a certain compression stress state. Modelling this phenomenon has been
the attractive subject of studies in many geotechnical applications, e.g. waste
deposition, soil and rock mechanics. Usually, such modelling is performed by
using rate-independent strain-softening plasticity models [1], which lead to ill-
posedness and mesh-dependent results [2, 3]. The mesh adaptive strategy is
essential for capturing the localization with the standard finite element method.

Recently, the strong discontinuity approach within the context of finite
element methods developed in [4, 5, 6, 7, 8] has been increasingly studied
and applied for modelling of localization phenomena and crack evolution [9,
10, 11, 12, 13, 14] due to the fact of its mesh-independence and its quadratic
convergence in the global Newton-Raphson scheme. Successful application
of this approach mainly requires: i) a bifurcation analysis corresponding to
the involved plasticity model, ii) an appropriate post-localization model to
represent the evolution of localization, iii) smooth function for enhanced strain
and iv) tracking the global discontinuity surfaces/paths to separate the failure
domain and therefore guarantee appropriate kinematics.

For localization problems in dilatant media (e.g. soil and rock), strong
discontinuity approaches have been successfully applied in a straightforward
manner in [11, 12] such that post-localization models arise naturally from the
corresponding associative plasticity model without introducing additional con-
stitutive law such as post failure models.

With bifurcation analysis [8], the information on the onset of discon-
tinuity in elements is attained. However, as a priori, the propagation of the
global discontinuity surface and/or path, which separates the failure domain
and, therefore, guarantees the appropriate kinematics, must be investigated in
order to make the strong discontinuity approach feasible. Two methods are
available to determine the global discontinuity surface:

**Element by element:** Starting from seed elements, the propagation of the
global discontinuity surfaces is determined by local tracking [15, 8].

**Global tracking:** The idea of the global tracking is that elements on the
global discontinuity surface are tracked at once using the feature of orien-
tations of all bifurcated elements. A successful realization of this scheme
is an algorithm based on the heat conduction equation [16], in which a
heat transport boundary value problem has to be solved, and further-
more the isolines of the quasi-temperature are the intended discontinuity
path.

This work is devoted to study the strong discontinuity approach with
enhanced strain element for modelling localization in dilatant media with non-
associative plasticity for plane strain problems and a priori, to develop an
algorithm to track the discontinuity evolution. Quadratic smooth functions are utilized for enhanced strain approximation to improve the poor accuracy in so called 'badly' tracked elements. The weak forms of the strong discontinuity approaches can be classified into different types symmetric statically consistent elements, symmetric kinematically consistent elements, non-symmetric statically and kinematically consistent elements [17, 13]. The last approach is adopted in the present study. Continuing the topic of post localization model, this work will address aspects of local tracking of discontinuity and propose an element level discontinuity tracking strategy. The algorithms provided are implemented in the object oriented scientific software package GeoSys/RockFlow [18, 19, 20, 21]. The newly developed methods of this work are tested with numerical examples for geomechanical problems, i.e. biaxial compression test and shear test.

2. Displacement with strong discontinuity

The enhanced strain field is based on the assumption that the strain localization exhibits a strong displacement discontinuity across the shear band [4]. This study is restricted within the framework of infinitesimal displacement field.

Consider the following displacement field, which has a jump across a discontinuity surface \( S \) in a bounded domain \( \Omega \)

\[
(2.1) \quad u = \tilde{u} + [u]H_S(x), \quad H_S(x) = \begin{cases} 
1 & \forall x \in \Omega_+ \\
0 & \forall x \in \Omega_- 
\end{cases},
\]

where \( \tilde{u} \) denotes the continuous part of displacement \( u \), \([u]\) denotes the displacement jump across \( S \) separating the continuous domain \( \Omega \) into subdomains \( \Omega_- \) and \( \Omega_+ \), \( H_S(x) \) is the Heaviside step function. Figure 1 gives a kinematic description of a strong discontinuity.

![Fig. 1. Strong discontinuity](image-url)
Introducing symmetric operators as [4]
\[ \nabla^s a|_{ij} = \frac{1}{2}(a_{i,j} + a_{j,i}), \]
\[ (a \otimes b)^s|_{ij} = \frac{1}{2}(a_i b_j + a_j b_i), \]
where \( a, b \in \mathbb{R}^n \). The strain rate tensor resulting from the jump displacement field is therefore given as
\[
\dot{\varepsilon} = \frac{1}{2} \left( \nabla \dot{u} + (\nabla \dot{u})^T \right) \\
= \nabla^s \dot{u} + (\dot{u} \nabla H_\mathcal{S}(x) + (\dot{u} \nabla H_\mathcal{S}(x))^T),
\]
where \( H_\mathcal{S}(x) \) is the heaviside function, the gradient of which has the form
\[
\nabla H_\mathcal{S}(x) = \delta_\mathcal{S}(x)n,
\]
with \( \delta \), the Kronecker delta function. The strain field (2.2) across \( \mathcal{S} \) can be written in a compact form as
\[
\dot{\varepsilon} = \nabla^s \dot{u} + \nabla^s [\dot{u}] H_\mathcal{S} + (\dot{u} \otimes n)^s \delta_\mathcal{S},
\]
where \( n \) is the normal to the discontinuity surface \( \mathcal{S} \).

### 3. Plasticity with strong discontinuity

We consider a non-associative elasto-plastic model defined by yield function \( f \) and plastic potential \( g \). The constitutive law and flow rule can be expressed as
\[
\hat{\sigma} = C^e(\dot{\varepsilon} - \dot{\varepsilon}^p),
\]
\[
\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma}, \quad \mathcal{H} \dot{\sigma} = -\dot{\lambda} \frac{\partial f}{\partial \sigma},
\]
Together with the unloading-loading Kuhn-Tucker criterion
\[
\dot{f} \leq 0, \quad \dot{\lambda} f = 0 , \quad \dot{\lambda} \geq 0,
\]
where \( \mathcal{H} \) is the isotropic hardening modulus, \( \lambda \) is the plastic multiplier and \( C^e \) is the 4th order elasticity tensor.
3.1. Localization condition

Denote the perfect plastic tangent modulus arising from (3.1) as \( C^{ep} \). The appearance of localization is characterized by \([22, 10]\)

\[
Q \cdot [\dot{u}] = 0, \tag{3.3}
\]

where \( Q \) is the acoustic tensor given as

\[
Q = n \cdot C^{ep} \cdot n. \tag{3.4}
\]

Equation (3.3) is the so called bifurcation condition.

3.2. Post localization plasticity model

In particular, attention should be paid to the stress-strain relationship, i.e. the constitutive law, when localization appears and plastic flow is localized to \( S \). Indeed, this constitutive law is characterized by the relationship between the traction on the discontinuity surface \( S \) and the displacement jump \([\dot{u}]\).

For a plane strain problem, only in-plane stresses, i.e. stresses excluding \( \sigma_{zz} \) come into the traction.

In this study the constitutive law is formulated within the framework of a non-associative plasticity. To this purpose, we assume that an associative flow rule is valid in the localized area, i.e. \( g = f \). Additionally, a scalar \( \lambda_\delta \) is introduced as the condition for the appearance of the localized plastic flow \([5]\)

\[
\dot{\lambda} = \dot{\lambda}_\delta \delta. \tag{3.5}
\]

Expression (3.5) indicates that the dissipation is independent of the geometry.

Considering the in-plane condition, we denote in-plane stress tensor as \( \sigma_i := \sigma_{ij} \) \((i, j = 1, 2)\) and have \( g = f = f(\sigma_i, q) \). In the localization shear band the rate of the plastic deformation can be decomposed to regular and singular parts as

\[
\dot{\epsilon}^p = \dot{\epsilon}^p + \dot{\lambda}_\delta \delta \frac{\partial f}{\partial \sigma_i}. \tag{3.6}
\]

For the regular part of expression (3.6), the non-associative plasticity is used and that leads to

\[
\dot{\epsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma}. \tag{3.7}
\]
From equations (3.1) and (3.5), the expression for the stress rate is obtained as

\[
\dot{\sigma}_i = C^e (\dot{\varepsilon} - \dot{\varepsilon}^p) + C^e \left[ (\dot{\mathbf{u}} \otimes \mathbf{n})^s - \lambda_\delta \frac{\partial f}{\partial \sigma_i} \right] \delta_{ij} .
\]

Since the stress must be regular, the singular part in equation (3.8) has to vanish, i.e. \((\dot{\mathbf{u}} \otimes \mathbf{n})^s - \lambda_\delta \frac{\partial f}{\partial \sigma_i} = 0\). This yields

\[
(\dot{\mathbf{u}} \otimes \mathbf{n})^s = \lambda_\delta \frac{\partial f}{\partial \sigma_i} .
\]

As that pointed in [23, 10], equation (3.9) would be invalid if the yield condition is characterized by total stress \(\mathbf{\sigma}\), e.g. \(f = f(\mathbf{\sigma}, q)\). The reason is there would be an inconsistency in rank such that \((\dot{\mathbf{u}} \otimes \mathbf{n})^s\) has two non-zero eigenvalues, while \(\lambda_\delta \frac{\partial f}{\partial \sigma_i}\) would have three eigenvalues even if for plane strain problems.

Usually, an extra constitutive law, or a damage model, is introduced for the post-localization state. On the contrary, in [11], a post localization softening law is derived naturally from the corresponding plasticity model. Although the coefficients of this softening law agree to the Mohr-Coulomb model, the inconsistency of the rank is not kept anymore.

To guarantee the rank consistency for plane strain problem, we assume that the post-localization stress-strain relationship is only governed by in-plane stress components. Consider a local coordinate system consisting of normal vector \(\mathbf{n}\) and \(\mathbf{t}\) the tangential to the internal discontinuity surface \(\mathscr{S}\) and cast expression (3.9) to develop a post localization model. The displacement jump rate can be expressed in the local coordinate system as

\[
[\dot{\mathbf{u}}] = \zeta_n \mathbf{n} + \zeta_t \mathbf{t},
\]

where \(\zeta_n\) and \(\zeta_t\) are the normal component and tangential component of displacement jump \([\mathbf{u}]\). Substituting equation (3.10) into equation (3.9) gives us

\[
\dot{\zeta}_n (\mathbf{n} \otimes \mathbf{n})^s + \dot{\zeta}_t (\mathbf{n} \otimes \mathbf{t})^s = \lambda_\delta \frac{\partial f}{\partial \sigma_i} .
\]

We denote in-plane identity as \(\mathbb{I} := \delta_{ij}, \ i, j = 1, 2\). The convolution product of equation (3.11) and the unit (identity) tensor \(\mathbb{I}\) gives

\[
\lambda_\delta \frac{\partial f}{\partial \sigma_i} : \mathbb{I} = \dot{\zeta}_n .
\]
We introduce an in-plane deviatoric operator as \( \text{dev} \ a = a - (a : I)I/2 \). Applying this operator on both sides of equation (3.9) results in

\[
\frac{1}{2}(\dot{\zeta}_n^2 + \dot{\zeta}_t^2) = \lambda_3^2 \left\| \text{dev} \ \left( \frac{\partial f}{\partial \sigma_i} \right) \right\|^2.
\]

Similarly, we define a localized dilatancy \( \Phi \) as [11]:

\[
\Phi = \frac{\dot{\zeta}_n}{|\zeta_t|}.
\]

Substituting equation (3.12) into equation (3.13) gives the expression of localized dilatancy as

\[
\Phi = \sqrt{\left( \frac{\partial f}{\partial \sigma_i} : I \right)^2 / 2 \left[ \frac{\partial f}{\partial \sigma_i} : \frac{\partial f}{\partial \sigma_i} - \left( \frac{\partial f}{\partial \sigma_i} : I \right)^2 \right]}.
\]

Since \( f = 0 \) holds for initial yield and post-yielding stages, we have a consistency condition as

\[
\dot{f} = \frac{\partial f}{\partial \sigma_i} : \dot{\sigma}_i + \frac{\partial f}{\partial q} : \dot{q},
\]

\[
= \frac{\partial f}{\partial \sigma_i} : \dot{\sigma}_i - \lambda (\frac{\partial f}{\partial q})^2 \mathcal{H} = 0.
\]

It is suggested that non-trivial value of \([u]\) requires \( \mathcal{H} \) taking the form as in [5]

\[
\mathcal{H}^{-1} = \mathcal{H}_\delta^{-1}\delta_{:Y},
\]

where \( \mathcal{H}_\delta^{-1} < 0 \), is a distributed non-singular function. Both the regular and the singular parts of the consistency condition (3.16) must be equal to zero if plastic deformation is localized on \( \mathcal{S} \). The condition of the singular part being zero is satisfied with equation (3.9) automatically. The zero condition of the regular part together with definition (3.17) result in

\[
\frac{\partial f}{\partial \sigma_i} : \dot{\sigma}_i = \lambda \delta \left( \frac{\partial f}{\partial q} \right)^2 \mathcal{H}_\delta.
\]

Applying convolution product of equation (3.11) with the in-plane stress rate tensor yields

\[
\dot{\zeta}_n \cdot \dot{t}_{:Y} + \dot{\zeta}_t \cdot \dot{t}_{:Y} = \lambda \delta \frac{\partial f}{\partial \sigma_i} : \dot{\sigma}_i,
\]
where $\mathbf{t}_\mathcal{S} = \mathbf{\sigma}_1 : \mathbf{n}$, is the traction vector on the discontinuity surface $\mathcal{S}$. Substituting expressions (3.12) and (3.18) into equation (3.19) leads to the post localization plastic model

$$
\Phi \mathbf{n} \cdot \mathbf{t}_\mathcal{S} + \text{sign}(\zeta_t) \mathbf{t} \cdot \mathbf{t}_\mathcal{S} = \left[ \Phi \frac{\partial f}{\partial q} / \left( \frac{\partial f}{\partial \sigma} : \mathbb{I} \right) \right]^2 \mathcal{H}_\delta|\zeta_t|.
$$

4. Non-associative Drucker-Prager plasticity

As a special case, we use the non-associative Drucker-Prager model to proceed the formulation given in Section 3. This model is usually applied to describe the behaviour of dilatant materials such as soil and rock. Denoting $\mathbf{I}$ as the identity tensor of order three and $\mathbf{s} = \mathbf{\sigma} - (\mathbf{\sigma} : \mathbf{I})\mathbf{I}/3$ as deviatoric stress, the yield function of the model reads

$$
f = \|\mathbf{s}\| + \alpha \mathbf{\sigma} : \mathbf{I} + q(\kappa) = 0,
$$

where $\alpha$ is an internal friction related material parameter. For non-associative plasticity, the plastic potential of this mode has the form

$$
g = \|\mathbf{s}\| + \beta \mathbf{\sigma} : \mathbf{I} + q(\kappa),
$$

with a dilatant associated material parameter $\beta$. $\kappa$ in both (4.1) and (4.2) is an internal variable. We use plastic strain definition for $\kappa$ as

$$
k = \dot{\varepsilon}^p = \sqrt{\frac{3}{2}} \int_0^t \|\mathbf{\dot{\varepsilon}}^p\| \, d\tau,
$$

where the time-like variable $t$ can be any monotonically increasing parameter controlling the loading process. As a consequence, we have

$$
q(\kappa) = \sqrt{\frac{2}{3}}(\sigma_0 + H \kappa)
$$

for linear hardening with initial yield stress $\sigma_0$ and hardening parameter.

A localization condition for this model has been derived from general condition (3.3) as [10]

$$
h = (\alpha - \beta)^2 - 2(1 - \nu) \left( \frac{s_{33}}{\|\mathbf{s}\|} + \frac{1}{2}(\alpha + \beta) \right)^2 = 0,
$$

where $s_{33}$ is the off in-plane component of the deviatoric stress $\mathbf{s}$. 

As pointed out in Section 3, the displacement-stress relationship in the strain localized area, or the post localization plastic model, are controlled by in-plane stress only. It is assumed that the material behaviour to be represented by associative plasticity model. We have in-plane plasticity model with

\[ f = g = ||\text{dev}\sigma_i|| + \alpha \sigma_i : \mathbb{I} + q(\kappa) = 0. \tag{4.6} \]

Before proceeding to the post localization plastic model, some mathematical expressions for the Drucker-Prager model are introduced herein. The derivatives of the yield function \( f \) with respect to stress \( \sigma_i \), are obtained as

\[ \frac{\partial f}{\partial \sigma_i} = \frac{\text{dev}\sigma_i}{||\text{dev}\sigma_i||} + \alpha \mathbb{I}. \tag{4.7} \]

This leads to

\[ \frac{\partial f}{\partial \sigma_i} : \frac{\partial f}{\partial \sigma_i} = 1 + \alpha^2, \quad \frac{\partial f}{\partial \sigma_i} : \mathbb{I} = 2\alpha. \tag{4.8} \]

Substituting equations (4.8) into (3.15) and (3.20), we attain the localized dilatancy

\[ \Phi = \alpha \sqrt{\frac{2}{1 - 3\alpha^2}}, \tag{4.9} \]

and the post localization plasticity model

\[ \Phi \mathbf{n} \cdot \mathbf{i}_\gamma + \text{sign} (\zeta_i) \mathbf{t} \cdot \mathbf{i}_\gamma = \frac{\mathcal{H}_d |\dot{\zeta}_i|}{2(1 - 3\alpha^2)}, \tag{4.10} \]

respectively for the Drucker-Prager model.

5. Enhanced finite element formulation

Since the plastic deformation is history dependent, we have to solve the problem in an incremental manner, i.e. loads and boundary conditions are imposed in the series of increments. The change of displacements and stresses are calculated during each successive increment. Assume the displacement \( u^n \) has already been obtained for the previous time step and \( \Delta u \) is the displacement increment of the current time step, then the solution of the current time step takes the form

\[ u^{n+1} = u^n + \Delta u, \]
where the superscripts $n$ and $n+1$ are the time step indices.

5.1. Enhanced element

For numerical simulation of the localized deformation with the finite element method, a continuum element with embedded discontinuities is used to establish the enhanced strain field.

We consider a regular element with an additional node, or an internal degree of freedom, whose associated shape functions are discontinuous inside the element. Assume that the regular element has $n_e$ nodes. The following discrete displacement approach is proposed in the incremental manner,

$$\Delta u = N_u \Delta \bar{u} + M_{\mathcal{S}} \alpha,$$

with shape function $N_u = [N_u^1 N_u^2 \cdots N_u^{n_e}]$, unknowns at element nodes $\Delta \bar{u} = [\Delta \bar{u}_1 \Delta \bar{u}_2 \cdots \Delta \bar{u}_{n_e}]^T$, additional node $\alpha = [\alpha_1 \alpha_2 \alpha_3]^T$ and the discontinuity shape function

$$M_{\mathcal{S}} = H_\mathcal{S} - \phi_e.$$

In (5.2), $\phi_e$ is an arbitrary smooth function that satisfies $\phi_e = 0$ in $\Omega^e_-$ and $\phi_e = 1$ in $\Omega^e_+$. Usually, linear smoothing functions are adopted which will prevent the discontinuity propagation if an element is badly tracked [9]. To improve this situation, we use quadratic finite elements and apply quadratic shape functions to smooth function $\phi_e$ regardless of element type as

$$\phi_e = \sum N_i^e, \forall i \big| x_i \in \Omega^e_+.$$

For 2D problems, we are going to prove a proposition for checking whether an element node is in $\Omega^e_+$ under the assumption that $\mathcal{S}_e \in \Omega^e$ is linear. With what depicted in Fig. 2, we have to prove the following proposition.

**Proposition 1.** Assume $x_{\mathcal{S}_e} \in \mathcal{S}_e$ is a point on element discontinuity line $\mathcal{S}_e$ and $n$ is the jump direction. A point $x_i \in \mathbb{R}^2$ is in domain $\Omega^e_+$ if and only if

$$(x_i - x_{\mathcal{S}_e}) \cdot n > 0.$$

**Proof.** Assuming the angle from $x_i - x_{\mathcal{S}_e}$ to $n$ is $\theta$, $(x_i - x_{\mathcal{S}_e}) \cdot n$ is then expressed as

$$(x_i - x_{\mathcal{S}_e}) \cdot n = |(x_i - x_{\mathcal{S}_e})||n| \cos \theta.$$
If inequality \((x_i - x_s) \cdot n > 0\) is satisfied by \(x_i\), we have \(\cos \theta > 0\) and this implies that \(x_i \in \Omega_+^e\) with \(\theta \in (-90^\circ, 90^\circ)\). □

**Remark.** The internal degree of freedom \(\alpha\) is the discretized form of the displacement jump \(\|\Delta u\|\).

For plane strain problem, \(\alpha = [\alpha_1, \alpha_2]\), the mapping of \(\alpha\) to the local coordinate system leads to

\[
\alpha = (n_t)[\Delta \zeta_n \Delta \zeta_l]^T.
\]

In the finite element formulation, strain tensor and stress tensor are usually given as two conjugated vectors. For plane strain problems, they are defined by

\[
\epsilon = [\epsilon_{11} \epsilon_{22} 0 \epsilon_{12}]^T, \quad \sigma = [\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12}]^T,
\]

for isotropic plane strain problems (hereafter, strain means engineering strain). The strain corresponding to the displacement (5.1) is therefore attained as

\[
\Delta \epsilon = \mathcal{L}\Delta u = \mathcal{L}(N_u \Delta \dot{u} + M_{\mathcal{X}}(n_t)[\Delta \zeta_n \Delta \zeta_l]^T)
\]

\[
= B\Delta \dot{u} + (P_e \delta_{\mathcal{X}} - G_e)[\Delta \zeta_n \Delta \zeta_l]^T,
\]

where \(\mathcal{L}\) is the linear strain operator, \(B = \mathcal{L}N_u\) is the linear strain-displacement matrix, \(P_e\) is a compact form for the singular part of the enhanced strain, and \(G_e\) is a compact form for the regular part of the enhanced strain. The definition of \(\mathcal{L}\), the expansions of \(P_e\) and \(G_e\) are given in Appendix.

### 5.2. Residual equations

In this work, the strong discontinuity approach is cast within the framework of non-symmetrical statically and kinematically consistent elements. Cor-
responding to the enhanced strain expression (5.4) and the weak form of momentum balance equation

\begin{equation}
\int_{\Omega} \hat{\sigma} : \nabla^s \eta \, d\Omega = \int_{\Omega} \hat{f} \cdot \eta \, d\Omega + \int_{\Gamma_t} \hat{t} \cdot \eta \, d\Gamma,
\end{equation}

a residual equation is derived by finite element discretization

\begin{equation}
r_e = \int_{\Omega^e} B^T \sigma^{n+1} \, d\Omega - \int_{\Omega^e} N_u f^{n+1} \, d\Omega - \int_{\Gamma_t} N_u \bar{t}^{n+1} \, d\Gamma = 0.
\end{equation}

In the displacement jump field, it is assumed that there is no any traction jump across the discontinuity surface \( \mathcal{S} \), which implies that

\[(\sigma_+^{n+1} - \sigma_-^{n+1}) \cdot n = 0.\]

Hence, a weak form of the traction continuity for the localized element can be expressed as \([4, 5]\)

\begin{equation}
\int_{\Omega^e_{loc}} \delta \tilde{\varepsilon} : \sigma_i^{n+1} \, d\Omega = 0,
\end{equation}

where \( \delta \tilde{\varepsilon} \) is the enhanced strain variation that corresponds to the piece-wise constant stress field. Two different formulae were proposed for the enhanced strain variation. The first one, which is usually applied, takes the form of \( P_e \) [4]. The second one incorporates the gradient of the post localization plasticity model with respect to stress [10]. Both of them show their ability in capturing localization. In the present study, the first one is adopted. This is

\begin{equation}
\delta \tilde{\varepsilon} = \left( \delta \mathcal{S} - \frac{l_{\mathcal{S}}}{A_e} \right) P_e [\delta \zeta_n \, \delta \zeta_t]^T,
\end{equation}

where \( l_{\mathcal{S}} \) is the length of the discontinuity within the element, and \( A_e \) is the area of the element. Since \( [\delta \zeta_n \, \delta \zeta_t]^T \) is an arbitrary function, we have

\begin{equation}
\int_{\Omega^e_{loc}} \left( \delta \mathcal{S} - \frac{l_{\mathcal{S}}}{A_e} \right) P_e^T : \sigma_i^{n+1} \, d\Omega = 0.
\end{equation}

Assuming the traction along the discontinuity \( \mathcal{S} \) within an element is constant, the singular part of (5.9) becomes

\begin{equation}
\int_{\Omega^e_{loc}} P_e^T : \sigma_i^{n+1} \, d\Omega = l_{\mathcal{S}} \int_{\mathcal{S}} P_e^T : \sigma_i^{n+1} \, d\Gamma = l_{\mathcal{S}} t_{\mathcal{S}}.
\end{equation}
Remark. \( t_\mathcal{S} \), the traction on \( \mathcal{S} \), is expressed in the local coordinate system and it has the form

\[
\begin{bmatrix}
  n \\
  t \\
\end{bmatrix} \cdot t_\mathcal{S} = 
\begin{bmatrix}
  n \\
  t \\
\end{bmatrix} \cdot t_\mathcal{S} =
\]

where \( t_\mathcal{S} = \sigma_i \cdot n \).

Substituting expression (5.10) into equation (5.9) leads to an element level residual equation

\[
(5.11) \quad r_{\mathcal{S}} = -\frac{1}{A_e} \int_{\Omega_{loc}^e} \mathbf{P}_e^T : \mathbf{\sigma}_t^{n+1} \, d\Omega + t_\mathcal{S} = 0.
\]

Likewise that presented in [11], residual equation (5.11) leads to a scalar one by taking into account the relationship between the two displacement jump components, i.e. the equations (3.14) and (3.15), as well as the expression of the post localization plastic model (3.20). This is done by applying a projection operator \( l = [\text{sign}(\Delta \zeta_t) \Phi 1]^T \) on both sides of residual equation (5.11), which gives

\[
(5.12) \quad r_{\mathcal{S}} = -\frac{1}{A_e} l^T \int_{\Omega_{loc}^e} \mathbf{P}_e^T : \mathbf{\sigma}_t^{n+1} \, d\Omega + s_\mathcal{S} = 0,
\]

where

\[
(5.13) \quad s_\mathcal{S} = \text{sign}(\Delta \zeta_t) \Phi \, n \cdot t_\mathcal{S} + t \cdot t_\mathcal{S}.
\]

Consequently, the enhanced strain (5.4) is allowed to be represented by a displacement jump component, e.g. \( \zeta_t \), with the projection operator \( l \). Therefore, the strain can be expressed as

\[
(5.14) \quad \Delta \epsilon = B \Delta \hat{u} + (\mathbf{P}_e \delta_\mathcal{S} - \mathbf{G}_e) \mathbf{l} \Delta \zeta_t.
\]

5.3. Return mapping scheme and linearization

We adopt the implicit method to integrate the stress (3.8) The linearization of unloading-loading Kuhn-Tucker criterion (3.2) reads

\[
(5.15) \quad f \leq 0, \quad \Delta \lambda f = 0, \quad \Delta \lambda \geq 0.
\]

Assume \( \sigma^n \) is the converged stress of the last time step. The computation of stress at the current time step begins with an elastic trial. This can be expressed as

\[
(5.16) \quad \sigma^{tr} = \sigma^n + D^e B \Delta \hat{u},
\]
where \( D^e \) is the matrix form of \( C^e \).

Using notation (5.13), the post localization plastic model (3.20) reads

\[
\dot{s}_\psi = \left[ \Phi \frac{\partial f}{\partial q} / \left( \frac{\partial f}{\partial \sigma_s} : I \right) \right]^2 \mathcal{H}_\delta \text{sign}(\dot{\zeta}_t) |\dot{\zeta}_t|
\]
\[
= \left[ \Phi \frac{\partial f}{\partial q} / \left( \frac{\partial f}{\partial \sigma_s} : I \right) \right]^2 \mathcal{H}_\delta \dot{\zeta}_t. \tag{5.17}
\]

The displacement jump is assumed progressing if the following conditions are satisfied

\[
\dot{r}_{e,\psi} > 0, \quad \dot{s}_\psi > \left[ \Phi \frac{\partial f}{\partial q} / \left( \frac{\partial f}{\partial \sigma_s} : I \right) \right]^2 \mathcal{H}_\delta \dot{\zeta}_t, \tag{5.18}
\]
and the stress and the displacement jump are updated as

\[
\sigma^{n+1} = \sigma^{tr} - D^e B l \Delta \zeta_t, \quad \zeta^{n+1} = \zeta^n + \Delta \zeta_t. \tag{5.19}
\]

Otherwise the stress and the displacement jump takes the form

\[
\sigma^{n+1} = \sigma^{tr}, \quad \zeta^{n+1} = \zeta^n. \tag{5.20}
\]

### 5.4. Linearization and condensation

The linearization of (5.6) and (5.12) yields

\[
\frac{\partial r_e}{\partial \Delta \dot{u}} \Delta \dot{u} + \frac{\partial r_e}{\partial (\Delta \zeta_t)} \Delta \zeta_t = -r_e, \tag{5.21}
\]

where

\[
\begin{align*}
\frac{\partial r_e}{\partial \Delta \dot{u}} &= \int_{\Omega^e} B^T D^e B \, d\Omega, \\
\frac{\partial r_e}{\partial (\Delta \zeta_t)} &= -\int_{\Omega^e} B^T D^e G_e l \, d\Omega, \\
\frac{\partial r_{e,\psi}}{\partial \Delta \dot{u}} &= -\frac{1}{A_e} I^T \int_{\Omega^e_{loc}} P_e^T D^e B \, d\Omega, \\
\frac{\partial r_e}{\partial (\Delta \zeta_t)} &= \frac{1}{A_e} I^T \int_{\Omega^e_{loc}} P_e^T D^e G_e l \, d\Omega + \frac{d s_{\psi}}{d \Delta \zeta_t}. \tag{5.22}
\end{align*}
\]
For the Drucker-Prager plasticity, $d_s / d(\Delta \zeta_t)$ is attained from equation (4.10) as

$$\frac{d_s}{d(\Delta \zeta_t)} = \frac{\mathcal{H}_d}{2(1 - 3\alpha^2)}.$$  

(5.23)

The enhanced parameter $\Delta \zeta_t$ is determined locally employing the staggered condensation procedure such that: At the element level, it is assumed $\Delta \hat{u}$ being known from the previous converged load step, and $\Delta \zeta_t$ is obtained by solving equation (5.21) locally using the Newton-Raphson method (cf. [12]).

6. Discontinuity path propagation

In this section, we present an element-by-element discontinuity propagation tracking algorithm and its implementation for 2D problems. The following data are required for tracking the discontinuity propagation:

1. A seed element to trigger the discontinuity propagation: For homogeneous stress state, any element that has edge on the domain boundary can be a seed element. Otherwise, the elements at which bifurcation occurs first are chosen as seed elements (Fig. 3). If more than one seed elements exist, multi-discontinuity paths have to be tracked.

[Seed element diagram]

Fig. 3. Seed element and departure point

2. Departure points of the discontinuity paths: Points on the edge of seed elements are selected. Normally, the center point of the boundary edge of seed element is an appropriation choice (Point P0 in Fig. 3).
3. One orientation of the discontinuity surface of the seed elements.

As a consequence, the propagation of discontinuity path is then checked element by element. For 2D problems, under the assumption of element-wise constant displacement jump, the tracking algorithm is to find a set of connected straight lines across bifurcated neighbouring elements. Assuming element \( k \) is the newly found bifurcated element (triangle/quadrilateral) that has discontinuity. To determine the discontinuity line in element \( k \), we have to know three variables, its orientation, its start and end points. The start point \( x_s^k \) is given by the last element having discontinuity. The discontinuity orientation is perpendicular to the jump directions \( n_k \), which can be determined by the post failure model. At the end, the end point can be obtained straightforwardly by searching the other cross point of the discontinuity line and edges of element \( k \). Figure 4 depicts triangle and quadrilateral elements having discontinuity in.

![Discontinuity path in elements](image)

Fig. 4. Discontinuity path in elements

The determination of discontinuity line turns into:

1. Choose a jump direction, or the orientation of the discontinuity line starting from \( x_s^k \). The tracking is conducted under the assumption that the global discontinuity line must be smooth. Therefore, the jump direction having the smallest angle between the neighbour jump direction \( n_{k-1} \) is used.

2. Find the out point of the discontinuity line of the element, \( x_e^k \), which is the intersection between the line and one element edge. The line is determined by the jump direction \( n_k \) and the in point \( x_s^k \).

Assuming \( x_a \) and \( x_b \) are two ends of an element edge, the coordinate of intersection \( x \) (see Fig. 5) can be obtained by solving the following linear
Fig. 5. Intersection of discontinuity line and element edge

equations (6.1):

\[
\begin{align*}
(x - x_k^e) \cdot n_k &= 0, & \text{discontinuity line}, \\
(x_b - x_a) \times (x - x_a) &= 0, & \text{edge line},
\end{align*}
\]

The possible number of the intersections might be as big as the number of element edges but only one intersection, i.e. \(x_k^e\), is on the element edge and it can be identified by the following axioms:

**Axiom 1.** Assume \(x \in \mathbb{R}^2\) is a point, \(n\) piece-wise straight lines \(S_i\) \((i = 1, \cdots, n)\) forms a convex polygon as \(\bigcup^n S_i\), \(A\) is the area of the polygon and \(A_i\) is the area of triangle formed by \(x\) and the two ends of line \(S_i\). Point \(x\) is located on line \(S_k\) if and only if the following condition is satisfied

\[
A = \bigcup_i A_i, \: i \neq k.
\]

**Axiom 2.** Under the same assumptions in Axiom 1, the area of polygon \(A\) can be calculated by moving \(x\) to an arbitrary position within the polygon and accumulating areas of triangles as

\[
A = \bigcup_i A_i.
\]

Figure 4 illustrates Axiom 1 in the cases of triangle and quadrilateral elements.
Now we start to introduce an element by element tracking strategy based on the two axioms. We assume $E_s = \bigcup_{i=1}^{k-1} E_i$ is the set of elements having corresponding discontinuity $S_i \in E_i$ in the sequence of discontinuity propagation. Obviously, $S_s = \bigcup_{i=1}^{k-1} S_i$ are made up of the global discontinuity line. The tracking is performed after each converged time step. If discontinuity element set $E_s$ is empty, the bifurcation status of all elements is checked. Otherwise, only the neighbour elements of the last element in the set $E_s$ are taken into account for bifurcation analysis. If there is an element found to be crossed by a discontinuity surface, the element is added to the set $E_s$. The tracking will be terminated after all discontinuity surfaces propagated to a boundary of the domain. The concept is implemented in the algorithm given in Table 1.

As mentioned in Table 1, the element $E_k$ has to be found and its discontinuity $S_k$ has to be located as well. The algorithm for such performances is summarized in the Table 2.

With the present algorithms, the propagation of discontinuity path through elements can be tracked by only searching recursively the neighbours of the last found element with discontinuity after the path is seeded. Furthermore, the orientation of discontinuity path is determined locally under the assumption that it changes smoothly between neighbour elements. The algorithms can be extended to 3D applications simply using the same nomenclature addressed in this section except departure point replaced with departure line.

7. Numerical tests

In this section three applications are presented which cover typical problems from geomechanics, such as biaxial and shear tests as well as footing problems. Both triangle and quadrilateral elements with quadratic shape functions are used in the present numerical tests. Two examples are presented to verify the present algorithms and test the quadratic smooth function $\phi_e$ given in equation (5.3). Through all calculations in the examples, the convergence of the global Newton-Raphson step is monitored by the residual norm error with a tolerance of $10^{-4}$. The appearance of discontinuity within an element is identified with the bifurcation condition (3.4) or $h = 10^{-4}$ (see equation (4.5)).

7.1. Plane strain biaxial test

First, we analyze a plane strain biaxial problem to verify the present algorithms. The phenomenon of strain localization is observed in such kind of test [24]. From the view point of bifurcation theory, strain localization is a bifurcation phenomenon, which takes place when the velocity field moves away
Table 1. Algorithm of element by element tracking

**Perform** following procedure after each converged time step

```plaintext
if no seed elements
    { check the bifurcation status of all elements
        using (3.3) or (4.5)
        if bifurcated elements are found
            set them as seed elements
    }
else
    { loop over seed element $E_s$
        { loop until break
            { get last element $E_l$ in $E_s$
                check bifurcation status
                if neighbour element $E$ is bifurcated
                    { if $E$ shares an edge or a vertex with $E_l$
                        and if $E_l \in E$ come cross the shared edge
                            or the vertex
                            { compute $\mathcal{S}_E \in E$ using algorithm in Table 2
                                with arguments $E_{k-1} = E_l$ and $E_k = E$
                                add $E$ to list $E_s$
                            }
                        if no neighbour element is bifurcated, **break**
                    }
            }
        }
    }
```

Table 2. Algorithm for tracking discontinuity at element level

1. **Determine** jump orientation of $\mathcal{S}_k$ as $n_k = \sup(n_{k-1} \cdot n_l^{i}, l = 1, \ldots, m)$
2. **Set** the out point of $\mathcal{S}_{k-1}$ as the in point of $\mathcal{S}_k$
   $$x_k = x_{k-1}$$
3. **Calculate** element area $A$
4. **Get** number of element edges $n_{edge}$
5. **For** $i=1$ to $n_{edge}$
   { **Calculate** the intersection $x$ using equation (6.1)
     if $x$ does not exists, **continue**
     set a variable $A_c = 0$
     **For** $j=1$ to $n_{edge}$
     { if $i=j$ **continue**
       **Calculate** area $A_j$ of triangle with vertexes $x, x_j^a$ and $x_j^b$
       $$A_c = A_c + A_j$$
     } if $A = A_c$, the out point is attained as $x_k^a = x$, **break**
   }
from the branch of continuous solutions and takes a new path of discontinuous solutions. This leads to mesh sensitive solutions with standard finite element method.

![Fig. 6. Plane strain biaxial test](image)

The set-up of the biaxial compression problem (Fig. 6) as well as the material parameters are the same as that proposed in [9]:

- The geometry of the specimen is simplified to a rectangle with size of 1 m $\times$ 3 m.
- The bottom of the specimen is placed on a horizontal roll supporter. While the top of it is only allowed a vertical down motion $u$. Both lateral sides are traction free.
- Non-associative flow rule is adopted for the Drucker-Prager model. The material parameters are given in Table 3.

This set-up will give a homogeneously distributed stress status, which will cause the localization onset on all elements simultaneously. However, natural geologic media are more or less heterogeneous. Therefore it is essential, to seed the discontinuity in one element prior to localization in order to trigger the propagation of discontinuity path. Hereafter, an element with an edge on left lateral side boundary and close to the top is set as a seed with 3% lower in initial stress $\sigma_0$ in this test.

The analysis of this problem concerns following cases.
Table 3. Material parameters of the plane strain biaxial test

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>kPa</td>
<td>$2 \times 10^4$</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>Parameter $\alpha$</td>
<td></td>
<td>0.233345 (30° friction angle)</td>
</tr>
<tr>
<td>Parameter $\beta$</td>
<td></td>
<td>0.141421 (16.53° dilatancy angle)</td>
</tr>
<tr>
<td>Initial stress $\sigma_0$</td>
<td>kPa</td>
<td>29.69 (20 of initial cohesion)</td>
</tr>
<tr>
<td>Hardening modulus $H$</td>
<td>kPa</td>
<td>100</td>
</tr>
<tr>
<td>Localized softening modulus $H_{\delta}^{-1}$</td>
<td>kPa</td>
<td>varying</td>
</tr>
</tbody>
</table>

**Mesh independency**

In the first case, the domain is discretized by different meshes with different element sizes and types. The vertical down motion is increased to 0.04 m in 20 time steps. The localization softening modulus $H_{\delta}^{-1}$ is $-1000$ kPa. The deformed mesh with contour plot of vertical displacement after 20 time steps are shown in Figs 7 and 8, respectively. Displacement is amplified with factor 10 for visualization purposes.

Figures 7 and 8 demonstrate that the localization is captured properly for different meshes, i.e. coarse Fig. 7(a) or fine Fig. 7(b), irregular Fig. 7 or regular Fig. 8. Moreover, the discontinuity patterns is very close to that observed in a test presented in [24]. Regardless which type of mesh is used either the mesh in Fig. 7 or in Fig. 8 for computation, the orientation of localization is obtained to be at an angle of 56° degrees with respect to the horizontal direction and the initial onset of discontinuity occurs when the vertical stress component $\sigma_{yy}$ is 73.40 kPa. These results are nearly the same as that obtained in [9]. Corresponding to $\sigma_{yy} = 73.40$ kPa, the vertical displacement is 0.0256 m. The variation of the vertical reaction on the top of the domain is shown in Fig. 9.

**Material of the post localization plasticity model**

The post localization plasticity model used for this analysis is the same as that presented in [11, 12] except for the material parameters. We use the grid depicted in Fig. 7(b) to test the influence of the material parameters. The localized softening modulus $H_{\delta}^{-1}$ is $-1000$ kPa. The variation of vertical reaction to the vertical down motion is shown in Fig. 10. Figure 10 implies that the present material parameter soften the medium a little after localization. The impact of variation of the post localization modulus $H_{\delta}^{-1}$ to the stability the strong discontinuity approach is also investigated. Cases with respectively $H_{\delta}^{-1} = -5\%$, 2.5\% and 1.5\% of Young’s modulus are examined. The results
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Fig. 7. Deformed irregular triangular grid

(a) 28 nodes, 36 elements  (b) 104 nodes, 168 elements

Fig. 8. Deformed regular grid: (a) triangle, (b) quadrilateral
Fig. 9. Vertical reaction of the top

Fig. 10. Vertical reaction of the top: Comparison with Ref. [11]
are depicted in the terms of the relationship between the vertical reaction and the vertical displacement in Fig. 11.

![Graph showing vertical reaction vs. vertical displacement](image)

**Fig. 11. Vertical reaction of the top: Comparison on varying of $H$**

Obviously, all the post-localization curves in Fig. 11 are straight lines as expected due to the constant $H^{-1}$. On the contrary, these corresponding curves presented in [9] are not straight line.

For all studied cases, the Newton-Raphson step converged quadratically within three iterations after onset of discontinuity. The results of this test portrays the stability and accuracy of the present algorithms as well as the capability of the present propagation tracking strategy.

### 7.2. Pure shear test

In this example, failure in a pure shear stress state in a rectangle domain of 2.5 m x 1 m (Fig. 12) is investigated. At the top of the domain, a horizontal displacement load is prescribed, while the movement in the vertical direction is fixed. On both of the left and right lateral sides, only the movement in horizontal direction is allowed. Displacements in any direction at bottom are fixed.

The plasticity of the material is assumed to obey non-associated plastic flow rule. Table 4 gives the material parameters.
With these material properties, the failure orientation can be obtained analytically as 84° from the horizontal direction. Three different meshes are used for this analysis, i.e. irregular triangulation, regular triangulation and quadrilateral meshing. A perturbation of initial yield stress is adopted for a selected element at the left boundary as seed of discontinuity. Figures 13, 14 and 15 show the propagation of the discontinuity surface for the different discretizations. The orientation of the discontinuity surfaces is very close to the analytical solution.

Figure 16 depicts the evolution of the shear stress along the increment of displacement load. The numerical simulations confirm that the localization occurs when the displacement load reaches 0.0025 m.

8. Conclusions

This paper is dealing with numerical modelling of discontinuity propagation in dilatant media representing geologic materials such as soil and rock. It is assumed that plasticity in these materials follows non-associative flow rule before the onset of failure. Afterwards, a post localization failure model is derived using the associative flow rule under the in-plane stress condition. Rank

Table 4. Material parameters of the pure shearing deformation problem

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
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<td>0.233345 (30° friction angle)</td>
</tr>
<tr>
<td>Parameter β</td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td>Initial stress σ₀</td>
<td>kPa</td>
<td>20.0</td>
</tr>
<tr>
<td>Hardening modulus H</td>
<td>kPa</td>
<td>0</td>
</tr>
<tr>
<td>Localized softening modulus H⁻¹</td>
<td>kPa</td>
<td>−1000</td>
</tr>
</tbody>
</table>
consistency is considered in the presented failure model. A quadratic smooth function is employed for enhanced strain finite elements to improve the kinematic properties of the elements with strong discontinuities. A new algorithm of this enhanced finite element formulation is outlined. An algorithm for efficient neighbour tracking starting from the last element of discontinuity onset is developed. As a result, a systematic approach in the framework of the enhanced strain finite element method for discontinuity propagation in dilatant media is presented. Two application examples representing typical problems in geomechanics are discussed to verify the presented algorithms for triangle and quadrilateral elements. First, a plane strain biaxial test for homogeneous stress status is analyzed. The results including the orientation of the discontinuity path agree very well with findings from [9]. Furthermore mesh independency
is proved. In the second test, the failure in a pure shear stress state is analyzed. The propagation of the discontinuity could be tracked very well even with a coarse mesh. Through these simulations, we found that the bifurcation condition is a key factor to ensure the propagation of the discontinuity surface. The discontinuity propagation could be captured successfully in both examples related to homogeneous and inhomogeneous stress states, respectively. Quadratic convergence is attained in all computations regardless of element type and mesh density. This demonstrates the stability and robustness of the presented strong discontinuity approach with enhanced finite elements. The tracking strategy for discontinuity propagation presented here has been shown its reliability through the present tests.

**Appendix**

Herein, all expressions are for 2D problems. The unit mapping vector
is defined as
\[ \mathbf{m}^T = [1, 1, 0] . \]

The linear strain operator is
\[ \mathcal{L} = \begin{pmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{pmatrix} . \]

Consider the normal direction \( \mathbf{n} \) and the tangential direction \( \mathbf{t} \) to the discontinuity surface \( \mathcal{S} \)
\[ \mathbf{n} = [n_1, n_2]^T, \quad [t_1, t_2]^T \]
and the expression (2.3). The singular part of the enhanced strain is derived as
\[
(7.1) \quad \mathbf{P}^e = \begin{pmatrix}
    n_1 n_1 & n_1 t_1 \\
    n_2 n_2 & n_2 t_2 \\
    2n_1 n_2 & n_1 t_2 + n_2 t_1
\end{pmatrix},
\]

Identically, the regular part of the enhanced strain is derived as
\[
(7.2) \quad \mathbf{G}^e = \begin{pmatrix}
    \frac{n_1}{\partial x_1} \frac{\partial \phi_e}{\partial x_1} & \frac{n_1}{\partial x_1} \frac{\partial \phi_e}{\partial x_2} & \frac{t_1}{\partial x_1} \frac{\partial \phi_e}{\partial x_1} \\
    \frac{n_2}{\partial x_2} \frac{\partial \phi_e}{\partial x_1} & \frac{n_2}{\partial x_2} \frac{\partial \phi_e}{\partial x_2} & \frac{t_2}{\partial x_2} \frac{\partial \phi_e}{\partial x_1} \\
    \frac{n_1}{\partial x_1} \frac{\partial \phi_e}{\partial x_2} + n_2 \frac{\partial \phi_e}{\partial x_1} & \frac{n_2}{\partial x_2} \frac{\partial \phi_e}{\partial x_2} + t_2 \frac{\partial \phi_e}{\partial x_1}
\end{pmatrix}.
\]

REFERENCES


