MODELLING OF FLUIDIZED GEOMATERIALS:
APPLICATION TO FAST LANDSLIDES*

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ABSTRACT. Fast catastrophic landslides cause many victims and important economic damage around the world every year. It is therefore important to predict their path, velocity and depth in order to provide adequate mitigation and protection measures. The distance travelled by these fluidized avalanches is large in many cases, such as lahars in volcanoes. Three dimensional models are extremely expensive, and depth integrated models provide a reasonable compromise between computational cost and accuracy. One important aspect to model is the constitutive/rheological behaviour of the materials. This paper describes both from the solid and from the fluid dynamics points of view models which can be used to describe fluidized soil behaviour. Concerning initiation or triggering, we describe generalized plasticity models for liquefaction and collapse of loose metaestable soils, as these mechanisms are found in many fast landslides. Once the soil has fluidized, we shall describe its rheological behaviour, giving details of how to obtain depth integrated rheological models.

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1. Introduction

Mathematical models have to be completed using constitutive or rheological models relating stress and strain tensors. In the case of solid soils, great effort has been devoted in the past decades to develop accurate constitutive models accounting for the more important aspects of soil behaviour, and today there is a wide choice between many elastoplastic, viscoplastic, hypoplastic, non linear incremental and generalized plasticity models, just to number some of them.

One important limitation is that, so far, no satisfactory model exists that is able to reproduce the behaviour of soil mixtures under the full range of strain rates which appear in fast slope movement problems. After liquefaction or fluidization has taken place, the soil behaves in a fluid like manner, and models used here are different from those used for reproducing the triggering mechanism. Indeed, both classes of models for solid or fluidized soils are often referred to as “constitutive equations” and “rheological models”, even if they provide in both cases relations between stress and strain tensors and their rates.

The purpose of this work is to present models able to describe –with different degree of approximation– the behaviour of fluidized soils. We shall begin discussing the process of fluidization or liquefaction, where a solid soil is transformed from solid to a fluid like material, and presenting some of the many available constitutive models which allow describing this phenomenon. In the second section, we shall describe a general framework within which rheological laws such as Newtonian, Bagnold or Bingham fluids flow models can be derived as particular cases.

One particular type of models which are widely used is the so called “depth integrated” model obtained from the general 3D models by integrating along depth. They require determination of “depth integrated stresses” and “basal friction forces”. Here we shall provide insight on how to derive both.

2. Mechanisms of fluidization of geomaterials
2.1. Types of failure and material behaviour

There are many alternative classifications of landslides attending to shape, morphology, velocity of propagation, etc. (Dikau et al. [6]). How-
ever, few of them provide insight on the mechanism of failure or the reasons why under some circumstances a landslide propagates with a reduced speed or it accelerates downhill, reaching high velocities and propagating over long distances.

Concerning initiation phase—or triggering mechanisms—, most classifications of landslides define two separate groups as “slides” and “flows”. In the former, a soil mass behaving like a rigid body moves along a clearly defined surface where strain localizes. This is the failure surface commonly used in classical geotechnical analysis, and it corresponds to a phenomenon of localization of plastic strain. In the latter failure takes place over a much larger extension of soil, and it is referred to as “diffuse failure”. Here it is important to mention the work of Darve and co workers in Grenoble, who proposed and investigated this mechanism (Darve and Laouafa [4]), and that of the authors (Pastor et al. [16]; Fernández-Merodo et al. [7]). Of course, nature is more complex, and we can find mixed types of landslides where liquefaction develops over a limited zone, or where water enters the soil during propagation of the landslide, etc. In most cases, fast catastrophic landslides are related to liquefaction of loose, poorly compacted deposits.

Therefore, we shall concentrate on “localized” and “diffuse” mechanisms, in order to relate them to the constitutive properties of the material.

Catastrophic landslides are characterized by an acceleration of the soil mass which is believed to be caused by a decrease of strength, which in turn is quite often related to “softening” of the material. However, it is important to precise what is really softening.

Let us consider the following three triaxial tests: (i) Consolidated Drained behaviour of a dense sand, (ii) Consolidated Undrained behaviour of a very loose sand, and (iii) Consolidated Undrained behaviour of an overconsolidated clay. Figure 1 illustrates these cases.

In the Figure 1, we have depicted: (a) deviatoric stresses versus axial strain, (b) stress paths on \( p' - q \) plane, and (c) the mobilized stress ratio \( (q/p') \) versus axial strain.

If we assume that soil strength is of frictional nature, it is the stress ratio \( (q/p') \) magnitude which we have to use in order to analyze soil response. In all the three cases depicted, we can see that only the CD response of dense sand and the CU response of the lightly overconsolidated clay present a decrease of strength. On the contrary, the CU response of the very loose sand is characterized by a monotonically increasing stress ratio. However, the peak in deviatoric strain is much more important in the latter. Therefore, lique-
faction is in many occasions responsible for diffuse failure mechanisms and fast catastrophic landslides while softening can be associated with localized mechanisms such as planar or circular slides.

Therefore, it is crucial to model the increase of pore pressure leading to liquefaction presented in loose or metaestable deposits. Liquefaction of loose sands has been studied in detail during the last decades, but the collapse of metaestable soils such as those of volcanic origin has received less attention from researchers. To illustrate the destructive power of landslides triggered on such soils, we can mention the case of El Salvador earthquakes of 2001, where a large number of fast landslides were triggered. In this case, the soils were unsaturated, and the long distances and velocities of propagation can be related to a collapse of soil structure causing a rise of the air pore pressure due to the tendency to compact. Of course, time of dissipation is small, but the propagation times were small than 10 s.
2.2. Constitutive modelling of liquefaction via Generalized Plasticity

There are excellent texts and state of art papers devoted to describe constitutive models and their use in geotechnical engineering. We can mention here the texts of Desai [5], Cambou and Di Prisco [3], Kolymbas [9, 10], Zienkiewicz et al. [20] among others, and the references provided therein.

We shall focus here on the Generalized Plasticity Theory (GPT) introduced by Zienkiewicz and Mroz [18] and elaborated by Zienkiewicz et al. [19] and Pastor et al. [14, 15], as it provides a framework within which accurate models can be developed to describe softening and liquefaction under monotonic and cyclic loading. The basic model has been recently extended by Tamagnini and Pastor [17] and Fernández Merodo et al. [7] to non saturated and collapsible soils.

Generalized Plasticity Theory introduces the dependence of the constitutive tensor relating increments of stress and strain on the direction of the increment of stress via a unit tensor $n$ which discriminates the states of “loading” and “unloading”

$$
\begin{align*}
    d\varepsilon &= C_L : d\sigma & \text{for } n : d\sigma^e > 0, \\
    d\varepsilon &= C_U : d\sigma & \text{for } n : d\sigma^e < 0,
\end{align*}
$$

where $d\sigma^e$ is the rate of stress which would be produced if the behaviour was elastic, $d\sigma^e = D^e : d\varepsilon$, and $D^e$ is the elastic constitutive tensor.

After imposing the condition of continuity between loading and unloading states, we arrive at

$$
\begin{align*}
    C_L &= C^e + \frac{1}{H_L} n_{gL} \otimes n, \\
    C_U &= C^e + \frac{1}{H_U} n_{gU} \otimes n.
\end{align*}
$$

In above, subindexes $L$ and $U$ refer to “loading” and “unloading”. The scalars $H_{L/U}$ are referred to as loading and unloading plastic moduli, and unit tensors $n_{gL/U}$ give the direction of plastic flow during loading and unloading.

The limit case, $n : d\sigma^e = 0$ is called “neutral loading”, and with the assumption done in (2), it can be seen that response is continuous as:

$$
\begin{align*}
    d\varepsilon_L &= C_L : d\sigma = C^e : d\sigma, \\
    d\varepsilon_U &= C_U : d\sigma = C^e : d\sigma.
\end{align*}
$$
The strain increment can be decomposed into two parts, elastic and plastic as:

\[ d\varepsilon = d\varepsilon^e + d\varepsilon^p, \]
\[ d\varepsilon^e = C^e : d\sigma, \]
\[ d\varepsilon^p = \frac{1}{H_L/U} n_g L/U \otimes n. \]

The main advantage of Generalized Plasticity Theory is that all ingredients can be postulated without introducing any yield or plastic potential surface. Moreover, it can be seen that both Classical Plasticity and Bounding Surface Plasticity models are special cases of the GPT.

We shall describe next a simple model proposed by Pastor et al. [14, 15] which is able to reproduce the basic features of sand behaviour under cyclic loading.

The main features of sand behaviour under monotonic and cyclic loading are the following:

(i) Volumetric deformations depend mainly on the stress ratio \( \eta = q/p \). There is a characteristic value \( \eta = M_g \) at which the behaviour changes from contractive to dilative. Failure at constant volume takes place also at this line, referred to as “Characteristic State Line” by Habib and Luong, and it can be interpreted as a Critical State Line for granular soils. The basic idea behind is that the soil, before failure, crosses a state at which there is no volume change, and comes back to it at residual conditions.

(ii) Very loose and loose sands exhibit compaction under shearing, which results in an increase of pore pressures when the loading process is not fully drained. In the limit, liquefaction can happen.

(iii) Dense sands exhibit dilation once the Characteristic State Line has been crossed. Dilation causes softening, and the strength decreases after a peak has been reached. Here, localization of strain in shear bands obscures the experimental results as the specimen is not homogeneous.

(iv) Under cyclic loading we observe the same compaction and dilation patterns. Plastic deformation occurs and the soil compacts progressively or the pore pressure increases. Liquefaction under cyclic loading is just the result of the increase of the pore pressure and the mechanism which is observed in monotonic loading.

(v) Medium dense sands under undrained cyclic loading develop a special type of behaviour which is referred to as ‘cyclic mobility’. The difference
to liquefaction consists in dilation which causes the pore pressure to decrease, hardening in turn the soil.

Taking into account all experimental facts described above, it is possible to develop a model within the Generalized Plasticity Theory as follows:

First of all, the direction of plastic flow in the \((p, q)\) plane is postulated as:

\[
\begin{align*}
n^T_{g} &= (n_{gv}, n_{gs}), \\
n_{gv} &= d_g/(1 + d_g^2)^{1/2}, \\
n_{gs} &= 1/(1 + d_g^2)^{1/2},
\end{align*}
\]

where the dilatancy \(d_g\) is given by \(d_g = (1 + \alpha) (M_g - \eta)\), \(\alpha\) is a material constant.

The loading-unloading discriminating relation \(n\) is obtained in a similar way:

\[
\begin{align*}
n^T &= (n_v, n_s), \\
n_v &= d_f/(1 + d_f^2)^{1/2}, \\
n_s &= 1/(1 + d_f^2)^{1/2}, \\
d_f &= (1 + \alpha) (M_f - \eta),
\end{align*}
\]

where \(M_f\) is a material constant.

The third ingredient is the plastic modulus, which has to be defined both for loading and unloading. During loading, we assume:

\[
H_L = H_0 p' H_f (H_v + H_s) H_{DM},
\]

where \(H_0\) is a constitutive parameter. In above, \(H_f\) is given by

\[
H_f = \left(1 - \frac{\eta}{\eta_f}\right)^4,
\]

and

\[
\eta_f = \left(1 + \frac{1}{\alpha}\right) M_f.
\]

\(H_f\) varies between 1 at \(q = 0\) to 0 at the straight line tangent to the Yield surface at the origin.
The terms $H_v$, $H_s$ and $H_{DM}$ refer, respectively, to volumetric and deviatoric strain hardening and the discrete memory. They are given by:

$$H_v = \left(1 - \frac{\eta}{M_g}\right),$$

(7)

$$H_s = \beta_0 \beta_1 \exp\left(-\beta_0 \xi\right),$$

$$H_{DM} = \left(\frac{\zeta_{\text{max}}}{\xi}\right)^\gamma.$$  

In above expression, $\beta_0$ and $\beta_1$ are material constants, while $\zeta$ characterized the maximum level of stress at which the material has been subjected. It is defined as

$$\zeta = p' \left[1 - \left(\frac{1 + \alpha}{\alpha}\right) \frac{\eta}{M}\right]^{1/\alpha}.$$  

Let us now consider each term. The volumetric term is zero at the CSL, and therefore, failure would take place there if $H_s$ were zero. It can be observed in triaxial tests that both in drained and undrained processes, the stress paths are able to cross this line. The role of $H_s$ is to prevent failure at this stage, but to allow it at residual conditions. This is achieved by making $H_s$ to depend on the accumulated deviatoric strain $\xi$ defined from $d\xi = (d\epsilon^p : d\epsilon^p)^{1/2}$ where $d\epsilon^p$ is the increment of the plastic deviatoric strain tensor.

As an example, we depict in Fig. 2 experimental results and model predictions for the liquefaction of very loose sand under undrained loading in a triaxial apparatus.

2.3. A Generalized Plasticity model for collapsible soils

An improvement of the Generalized Plasticity model has been recently proposed by the authors to reproduce the mechanical behaviour of bonded soils, weak rocks and other materials of a similar kind.

Following the framework introduced by Gens and Nova [8] and Lagioia and Nova [11], two basic concepts lie in the representation of this mechanical behaviour: the fundamental role played by yield phenomena and the need for considering the observed behaviour of the bonded material in relation to the behaviour of the equivalent unstructured one.

As the amount of bonding increases the yield surface must grow up. Two parameters define the new enlarged yield locus: $p_{c0}$ that controls the
yielding of the bonded soil in isotropic compression and $p_t$ which is related to the cohesion and tensile strength of the material. Both $p_{c0}$ and $p_t$ increase with the magnitude of bonding.

We can assume that the degradation of the material (decrease in bonding) is related to some kind of damage measure, that will in turn depend on plastic strains. Lagioia and Nova [11] proposed simple laws to describe the debonding effect on a calcarenite material. The evolution of $p_t$ is governed by:

$$p_t = p_{t0} \exp \left(-\rho_t \varepsilon^d \right),$$

where $p_{t0}$ and $\rho_t$ are two constitutive parameters and $\varepsilon^d$ is the accumulated plastic volumetric strain. It appears reasonable to assume that changes of the yield locus will be controlled by two different phenomena: conventional plastic hardening (or softening) for an unbonded material and bond degradation. In that case, the plastic modulus in the sand model proposed by Pastor et al. [15], can be modified introducing $H_b$ such as:

$$H_L = (H_{0p}^* - H_b) H^*_f (H^*_v + H_s) H^*_D M,$$
where the following notations are used:

\[
\begin{align*}
\eta^* &= q/(p' + p_t), \\
\zeta^* &= (p' + p_t) \left\{ 1 - \left( \frac{\alpha_f}{1 + \alpha_f} \right) \frac{\eta^*}{M^*} \right\}^{-1/\alpha}, \\
H_b &= b_1 \varepsilon^d \exp \left( -b_2 \varepsilon^d \right), \\
\end{align*}
\]

and \( \alpha_f, b_1, b_2, \gamma_{DM} \) and \( M \) are model parameters. It can be seen that the value of \( H_b \) decreases when the volumetric plastic strain increases (i.e. when debonding occurs) and in the limit case, when de-structuring is complete, \( H_b \) becomes zero. In this case, the modified plastic modulus defined above coincides with the original plastic modulus.

With this improvement it is possible to reproduce the laboratory tests of Lagioia and Nova [11] on the Gravina calcarenite. Figures 3 to 6 compare experimental data and model predictions for isotropic compression test and shearing under drained conditions with initial isotropic confining pressures as well as for oedometer test.
Fig. 4. Drained constant cell pressure test ($\sigma'_c = 1300$ kPa) experimental data from Lagioia and Nova [11] and model predictions.

Fig. 5. Drained constant cell pressure test ($\sigma'_c = 2000$ kPa) experimental data from Lagioia and Nova [11] and model predictions.
Fig. 6. Oedometer test experimental data from Lagioia and Nova [11] and model predictions

reflects a mechanism which in our opinion plays a paramount role in the generation of pore pressures and catastrophic failure of soils.

2.4. Comments and remarks

So far, we have considered behaviour of soil up to failure that is the point at which classical geotechnical analysis stops. Beyond it, there are important difficulties from all points of view: experimental, constitutive, and numerical. Experimentalists find that specimens are not homogeneous in the case of localized failure, and therefore, it is a difficult task to obtain reliable measurements of stress and strain. Moreover, the test apparatuses are not designed to follow the important strain which develops after failure. Constitutive modellers do not go further because of the lack of experimental data. The situation is not much different for numerical analysts, who will find here large deformations. Numerical methods such as finite elements need to use special techniques. If the formulation is of Lagrangian type, the mesh can be severely distorted, and either adaptive remeshing or arbitrary Lagrangian-Eulerian techniques have to be used. If an Eulerian formulation is used, the material movement has to be tracked in the fixed mesh. Tracking of free surfaces requires also the use of techniques such as the level set.

Concerning constitutive modelling—to which this paper is devoted—, one important aspect is to bridge the gap between solid and fluid behaviour.
Viscoplastic models can provide this bridge, if some concepts such as critical state are extended to the fluid phase.

The paradigm of the existence of a Critical State (CS) at which the material can sustain deformation at constant void ratio has to be revised. One possible solution could be to assume that the CS depends either on a measure of the rate of deformation or the kinetic energy of the material. The resulting state surface in the space kinetic energy - hydrostatic pressure and void ratio could explain some experimental observations in rheometers working at constant pressure or constant volume.

3. Rheological models

3.1. General framework

Once downhill movement has started, material behaviour becomes more and more “fluid-like”. The problem of modelling such fluid is complex, and it will depend on the type of mixture. In the simplest case of a flow of a granular material without interstitial water, it could be thought of being composed of a single phase. However, the flow presents inverse segregation, and the coarsest fraction will move upwards. Therefore, this phenomenon cannot be modelled if the assumption of a single phase material is made. The problem gets more complicated if an interstitial fluid is present. In some cases the pore fluid is water, while in others is mud.

A first simplification commonly found in literature consists in studying the overall mixture behaviour, formulating ad-hoc models for it. This approach precludes the relative movement of the interstitial fluid against the solid fraction, and does not allow modelling of stabilization of the flow using bottom drainage systems.

Next refinement consists in considering two phases, a granular skeleton and the fluid (water or mud) filling its voids. If the shear resistance of the fluid phase can be neglected, the stress tensor in the mixture can be decomposed into a “pore pressure” and an effective stress.

The purpose of this Section is to present some of the models which have been used in the past, providing values of their material parameters whenever they are available. Most of these models have been formulated in terms of total stresses for one phase, but they can be generalized including the stresses within the fluid. It seems reasonable to consider one phase in the case of mudflows, and two phases in the case of debris flows. The difference will be in the ability of the fluid phase to percolate through the solid.
One of the main difficulties to develop suitable rheological models is the limitations of the existing rheometers in applying general stress and rate of deformation fields. This situation is much different from that found in soil mechanics, where today it is possible to control three independent magnitudes such as principal stresses or invariants. There exist devices allowing to explore the effect of rotation of principal stresses axes, the structural and induced anisotropy, etc.

In the case of fluidized geomaterials, the stress conditions are rather simple, such as the simple shear which has been already described, and most of simplifications used in the analysis are caused by a lack of available data rather than provoked by findings in experiments.

If we assume that the fluid is isotropic, and we want to express the stress as a function of the rate of deformation tensor, it is possible to use the so called “representation theorems”. Following Malvern [12] the stress tensor can be expressed as:

\[
\sigma = -pI - \Phi_0 I + \Phi_1 d + \Phi_2 d^2,
\]

where \( p \) is a “thermodynamic” pressure, \( I \) the identity tensor, \( d \) the rate of deformation tensor, and \( \Phi_k, k = 0..2 \) scalar functions of the invariants of \( d \):

\[
\Phi_k = \Phi_k (I_{1d}, I_{2d}, I_{3d}).
\]

The invariants are defined as:

\[
I_{kd} = \frac{1}{k} \text{tr} \left( d^k \right).
\]

In most of models it is assumed that the fluid is incompressible, and therefore \( I_{1d} = 0 \). When modelling fast landslides, some authors consider the deformation decomposed into two parts: propagation, at constant volume, and vertical consolidation. In this case, using the assumption that behaviour of the mixture is incompressible is justified. However, the reader should be aware that this is just an assumption which has been proven accurate only under certain restrictions. We shall assume in what follows that the flow is incompressible. Moreover, due to the lack of experimental evidence, we shall assume that there is no dependence on \( I_{3d} \). In our opinion, this assumption which is found in most of the available models should be given more consideration. In the case of soils and granular materials in solid state, this assumption leads to important
errors, such as different friction angles at critical state at compression and extension. In consequence, we shall assume:

\[ \Phi_k = \Phi_k(I_2 d). \]

The stress tensor can be decomposed into hydrostatic and deviatoric components as:

(12) \[ \sigma = -\hat{p}I + s, \]

where:

\[ \hat{p} = -\frac{1}{3} \text{tr} (\sigma), \]

\[ s = \sigma + \hat{p}I. \]

The hydrostatic component is obtained as:

(13) \[ \hat{p} = -\frac{1}{3} (-3p - 3\Phi_0 + \Phi_2 \text{tr} (d^2)) \]

\[ = p + \Phi_0 - \frac{2}{3} \Phi_2 I_{2d}. \]

Finally, the deviatoric component can be easily obtained as:

(14) \[ s = -\frac{2}{3} \Phi_2 I_{2d} I + \Phi_1 d + \Phi_2 d^2. \]

It is important to notice that, as the flow is incompressible, the hydrostatic component of the rate of deformation tensor is zero, and \( d \) is a deviatoric tensor. Therefore,

(15) \[ d = \text{dev} (d), \]

and (14) can be written as:

(16) \[ s = -\frac{2}{3} \Phi_2 I_{2d} I + \Phi_1 \text{dev} (d) + \Phi_2 d^2. \]

The stress is given by:

(17a) \[ \sigma = -pI - \Phi_0 I + \Phi_1 \text{dev} (d) + \Phi_2 d^2. \]

A third alternative is:

(17b) \[ \sigma = -pI - \Phi_0 I + \Phi_1 \text{dev} (d) + \Phi_2 \{\text{dev} (d)\}^2. \]
We shall consider next two particular cases of motion: simple shear flow and a general 2D plane flow.

The simple shear flow will be assumed to take place in the $X_1 X_3$ plane, with a velocity field of the form:

\begin{equation}
\begin{aligned}
v_1 &= v_1(x_3), \\
v_2 &= v_3 = 0.
\end{aligned}
\end{equation}

The rate of deformation tensor is:

\begin{equation}
d = \begin{pmatrix}
0 & 0 & \frac{1}{2} \frac{\partial v_1}{\partial x_3} \\
0 & 0 & 0 \\
\frac{1}{2} \frac{\partial v_1}{\partial x_3} & 0 & 0
\end{pmatrix},
\end{equation}

and $d^2$ is given by:

\begin{equation}
d^2 = \begin{pmatrix}
\frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} \right)^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} \right)^2
\end{pmatrix}.
\end{equation}

The second invariant $I_{2d}$ is:

\begin{equation}
I_{2d} = \frac{1}{2} \text{tr} \left( d^2 \right) = \frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} \right)^2.
\end{equation}

The stress tensor is given by:

\begin{equation}
\sigma = -pI - \Phi_0 I + \Phi_1 \begin{pmatrix}
0 & 0 & \frac{1}{2} \frac{\partial v_1}{\partial x_3} \\
0 & 0 & 0 \\
\frac{1}{2} \frac{\partial v_1}{\partial x_3} & 0 & 0
\end{pmatrix}
+ \Phi_2 \begin{pmatrix}
\frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} \right)^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} \right)^2
\end{pmatrix},
\end{equation}
from where we obtain the components:

\[
\begin{align*}
\sigma_{11} &= \sigma_{33} = -p + \Phi_0 + \frac{1}{4} \Phi_2 \left( \frac{\partial v_1}{\partial x_3} \right)^2, \\
\sigma_{22} &= -p - \Phi_0, \\
\sigma_{13} &= \sigma_{31} = \frac{1}{2} \Phi_1 \frac{\partial v_1}{\partial x_3}.
\end{align*}
\]

(21)

It is important to notice that, in addition to having shear stresses which depend on the rate of shear strain \( \frac{\partial v_1}{\partial x_3} \) we find that the normal stresses \( \sigma_{11} \) and \( \sigma_{33} \) depend also on it. This contribution is often referred to as “dispersive stresses”.

The expression of the stress components (21) will be used to generalize the results obtained in simple shear flow rheometers to more general stress conditions.

The second case which will be considered here is that of a incompressible 2D flow, which we shall assume to take place in the plane \( X_1 X_3 \). The rate of deformation tensor is now given by:

\[
d = \begin{pmatrix}
    d_{11} & 0 & d_{13} \\
    0 & 0 & 0 \\
    d_{31} & 0 & d_{33}
\end{pmatrix}
\]

(22a)

with \( d_{11} + d_{33} = 0 \). From here, we obtain:

\[
d^2 = \begin{pmatrix}
    d_{11}^2 + d_{13}^2 & 0 & d_{13} (d_{11} + d_{33}) \\
    0 & 0 & 0 \\
    d_{13} (d_{11} + d_{33}) & 0 & d_{33}^2 + d_{13}^2
\end{pmatrix}
\]

(22b)

\[
= \begin{pmatrix}
    d_{11}^2 + d_{13}^2 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & d_{13}^2 + d_{33}^2
\end{pmatrix}.
\]

The components of the stress tensor are given now by:

\[
\sigma = -pI - \Phi_0 I + \Phi_1 \begin{pmatrix}
    d_{11} & 0 & d_{13} \\
    0 & 0 & 0 \\
    d_{31} & 0 & d_{33}
\end{pmatrix} + \Phi_2 \begin{pmatrix}
    d_{11}^2 + d_{13}^2 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & d_{33}^2 + d_{13}^2
\end{pmatrix},
\]

(23)
from where we obtain:

\[
\begin{align*}
\sigma_{11} &= -p - \Phi_0 + \Phi_1 d_{11} + \Phi_2 \left( d_{11}^2 + d_{13}^2 \right), \\
\sigma_{33} &= -p - \Phi_0 + \Phi_1 d_{33} + \Phi_2 \left( d_{33}^2 + d_{13}^2 \right) \\
&= -p - \Phi_0 - \Phi_1 d_{11} + \Phi_2 \left( d_{11}^2 + d_{13}^2 \right), \\
\sigma_{22} &= -p - \Phi_0, \\
\sigma_{13} &= \sigma_{31} = \Phi_1 d_{13}, \\
\sigma_{12} &= \sigma_{23} = 0.
\end{align*}
\]  

(24)

Again, we find (i) a shear stress depending only on \(d_{13}\), and (ii) dispersive stress terms which depend on \(\Phi_2 \left( d_{11}^2 + d_{13}^2 \right)\).

3.2. Bagnold fluids

Bagnold [1, 2] proposed a model valid for simple shear conditions which explained the behaviour observed in his rheometer. The tests were conducted at constant volume, and he identified two different flow regimes, which he denoted as “macro-viscous” and “inertial”, depending on whether fluid viscosity or grain inertia dominated. The transition was characterized by a value of the non dimensional number \(N_B\) in the range 40–450. The Bagnold number \(N_B\) is defined as:

\[
N_B = \frac{\lambda^{1/2} \rho c^2}{\mu} \left( \frac{\partial v_1}{\partial x_3} \right),
\]

(25)

where \(c\) is the diameter of the spheres, \(\rho\) is their density, and \(\lambda\) is the linear concentration.

Bagnold proposed the following expressions for shear and normal stresses in the inertia regime:

\[
\begin{align*}
\sigma_{13} &= a_i \sin \alpha_i \rho \lambda^2 c^2 \left( \frac{\partial v_1}{\partial x_3} \right)^2, \\
\sigma_{33} &= p + a_i \cos \alpha_i \rho \lambda^2 c^2 \left( \frac{\partial v_1}{\partial x_3} \right)^2,
\end{align*}
\]

(26)

where \(a_i = 0.042\), \(\tan \alpha_i = 0.4\). The equations are valid for \(\lambda \geq 12\).

Equations (26) show a dependence of the normal stress \(\sigma_{33}\) on the rate of shear deformation \(\frac{\partial v_1}{\partial x_3}\). This component is referred to as “dispersive
pressure”, and it can be observed under isochoric conditions. Should the test have been run under constant pressure conditions, we would have observed a dilation or increase of void ratio.

If we compare expressions (26) to equation (20), we can identify:

\[
\sigma_{33} = -p - \Phi_0 + \frac{1}{4} \Phi_2 \left( \frac{\partial v_1}{\partial x_3} \right)^2 = -p - a_i \cos \alpha_i \rho \lambda^2 c^2 \left( \frac{\partial v_1}{\partial x_3} \right)^2,
\]

\[
\sigma_{13} = \sigma_{31} = \frac{1}{2} \Phi_1 \frac{\partial v_1}{\partial x_3} = \sigma_{13}^{\text{turb}} + a_i \sin \alpha_i \rho \lambda^2 c^2 \left( \frac{\partial v_1}{\partial x_3} \right)^2,
\]

from where it follows:

\[
\Phi_0 = 0,
\]

\[
\Phi_1 = 2 a_i \sin \alpha_i \rho \lambda^2 c^2 \left( \frac{\partial v_1}{\partial x_3} \right) = 4 a_i \sin \alpha_i \rho \lambda^2 c^2 \left( I_{2D} \right)^{1/2},
\]

\[
\Phi_2 = -4 a_i \cos \alpha_i \rho \lambda^2 c^2.
\]

In above, we have used the value of \( I_{2D} \) for simple shear flows given in (19c).

From here, we can derive the general rheological model or the Bagnold fluid model:

\[
\sigma = -p I + \left\{ 4 a_i \sin \alpha_i \rho \lambda^2 c^2 \left( I_{2D} \right)^{1/2} \right\} \text{dev} (d) - \left\{ 4 a_i \cos \alpha_i \rho \lambda^2 c^2 \right\} d^2.
\]

Again, it is important to remark that Bagnold model has been obtained for a granular fluid under constant volume conditions, and special care should be taken when generalizing it to different situation.

4. Depth integrated Rheological Models

4.1. Introduction

Fast landslides, avalanches and debris flows are complex 3D phenomena which can be described by alternative models with different accuracy. In the more general situation, the equations of balance of linear momentum for all components of the mixture include the partial stresses, which have to be specified from the flow kinematics. In the preceding Section we have presented several rheological models which allow determination of the stress tensor once the rate of deformation tensor is known.

This approach is complex and expensive in computer time. As an alternative, we have introduced the depth integrated models, where velocities
and stresses are integrated along the vertical axis. The equations are cast in terms of depth of flow and averaged velocities and stresses, and include the surface forces on the surface and the bottom.

We shall recall here for completeness the depth integrated balance of momentum equations which describe the different alternative mathematical models available to describe fast landslides.

The basic equation is:

\[
(29a) \quad \frac{\partial}{\partial t} (h \bar{v}_i) + (1 + \alpha) \frac{\partial}{\partial x_j} (h \bar{v}_i \bar{v}_j) = b_i h + \frac{\partial}{\partial x_j} (h \bar{\sigma}_{i,j}) + |N^A| t^A_i + |N^B| t^B_i
\]

or the alternative form of it

\[
(29b) \quad \frac{\partial}{\partial t} (h \bar{v}_i) + \frac{\partial}{\partial x_j} \left( h \bar{v}_i \bar{v}_j - \frac{1}{2} \rho b_3 h^2 \delta_{ij} \right)
= \alpha \frac{\partial}{\partial x_j} (h \bar{v}_i \bar{v}_j) + \frac{\partial}{\partial x_j} (h \bar{\sigma}^*_i) + b_i h + |N^A| t^A_i + |N^B| t^B_i.
\]

It is important to notice:

(i) The forces at the basal surface $t^B_i$, which depend on the stresses.

(ii) The correction factor $\alpha$ defined from:

\[
(30) \quad \int_z^{z+h} v_i^* v_j^* dx_3 = \alpha \bar{v}_i \bar{v}_j,
\]

where $v_i^*$ is related to the depth average velocity by:

\[
v_i(x_1, x_2, x_3) = \bar{v}_i(x_1, x_2) + v_i^*(x_1, x_2, x_3).
\]

(iii) The depth integrated stress tensors $\bar{\sigma}_{ij}$ and $\bar{\sigma}^*_{ij}$ which have been obtained using the decomposition:

\[
(31a) \quad \bar{\sigma}_{ij} = -p^* \delta_{ij} + \bar{\sigma}^*_{ij},
\]

where

\[
p^* = \rho b_3 (Z + h - x_3).
\]

After averaging along depth, we arrive at:

\[
(31b) \quad \bar{\sigma}_{ij} = -\bar{p} \delta_{ij} + \bar{\sigma}^*_{ij}
\]
with
\[ \bar{p} = \frac{1}{2} \rho b_3 h, \]
\[ (31c) \]
\[ \bar{\sigma}_{ij}^3 = \int_{z}^{z+h} \sigma_{ij}^3 dx_3. \]

The original 3D equations have been greatly simplified as the resulting model is 2D, but the problem now is how to determine the correction factor defined in (30), the basal forces per unit area \( t_i^B \) and the depth integrated stresses (31b) without having information of the 3D flow structure, which has been lost in the averaging process.

A possible solution which is widely used consists of assuming that the flow at a given point and time, with known depth and depth averaged velocities has the same vertical structure as a uniform, steady state flow. In the case of fast landslides this model is often referred to as the infinite landslide, as it is assumed to have constant depth and move at constant velocity along a constant slope.

In addition to provide depth integrated stresses and basal friction, the infinite landslide model can be used in order to (i) determine the conditions at which a landslide is triggered, and (ii) provide a first estimate of paths and velocities.

The next subsection presents the infinite landslide model and the method to obtain basal friction and depth averaged stresses.

**4.2. The Infinite Landslide model**

In what follows, we shall assume that our infinite landslide takes place along a plane with constant slope. We shall use the notation introduced in Fig. 7, where the main variables which will be used in the analysis are depicted.

In the figure, we have considered a column of material of unit length and depth \( h \). The flow structure is that of a simple shear flow. As the acceleration along \( x \) axis is zero, we can write the equilibrium of the whole column as:

\[ \rho g h \sin \theta = \tau_B, \]
\[ (32) \]

where \( \rho \) is the mixture density, \( g \) the gravity acceleration, \( \theta \) the slope and \( \tau_B \) is the bottom friction. By considering the equilibrium of a part of the column extending from the surface to a depth \((h - z)\), we can obtain the shear stress as a function of \( z \):

\[ \tau = \rho g (h - z) \sin \theta = \tau_B \left( 1 - \frac{z}{h} \right). \]
\[ (33) \]
Therefore, the shear stress varies linearly from zero at the surface to a maximum at the bottom given by equation (32). This shear stress can be related to the rate of shear strain using the models described in the preceding Section. The ordinary differential equation can be integrated to obtain the velocity profile, from which we can determine the depth averaged velocity. Finally, the shear stress at the bottom will be derived eliminating the slope angle. We shall apply this method to the rheological models described in the preceding Section.

Once we have obtained the velocity $v(\xi)$ where $\xi = \frac{z}{h}$, we can write the velocity as a function of the average velocity:

\begin{equation}
    v(\xi) = \bar{v} f(\xi),
\end{equation}

where $f(\xi)$ is a function which depends on the rheological model. From here, we have: $v(\xi) = \bar{v} + v^*(\xi)$ and

\begin{equation}
    v^*(\xi) = \bar{v} (f(\xi) - 1).
\end{equation}

To obtain $\alpha$ from (30), we use the above equation for the components of velocity:

\begin{align*}
    v_i^* &= \bar{v}_i (f(\xi) - 1), & v_j^* &= \bar{v}_j (f(\xi) - 1),
\end{align*}
from where we arrive at:

\[ \alpha = \frac{\int_z^{z+h} v_i^* v_j^* \, dx_3}{\bar{v}_i \bar{v}_j} = \int_0^1 (f(\xi) - 1)^2 \, d\xi. \tag{35} \]

Therefore, the correction factor \( \alpha \) can be obtained from the information provided by the infinite landslide model.

Finally, we shall analyze the depth averaged stresses. Our purpose is to relate them to a depth integrated rate of deformation tensor. The starting point is the general rheological law (17).

If we average both sides of equation (17) along depth, the averaged magnitudes will not satisfy the averaged equation, unless the equation is linear, which only take place if \( \Phi_0 \) and \( \Phi_2 \) are zero and \( \Phi_1 \) is constant, i.e.

\[ \bar{\sigma} \neq -\bar{p}I + \bar{\Phi}_0I + \bar{\Phi}_1 \text{dev}(\bar{d}) + \bar{\Phi}_2 \bar{d}^2. \]

If the conditions for linear relation in (17) are satisfied, we can write:

\[ \bar{\sigma} = -\bar{p}I + \Phi_1 \text{dev}(\bar{d}). \]

To obtain the averaged components of the rate of deformation tensor \( d_{ij} \), we shall first consider the case \( i, j = 1, 2 \), thus:

\[ \bar{d}_{ij} = \frac{1}{2h} \left( \int_z^{z+h} \frac{\partial v_i}{\partial x_j} \, dx_3 + \int_z^{z+h} \frac{\partial v_j}{\partial x_i} \, dx_3 \right). \tag{36} \]

In the case of the simple shear flow or infinite landslide the integral \( \frac{1}{h} \int_z^{Z+h} \frac{\partial v_i}{\partial x_j} \, dx_3 \) is evaluated as:

\[ \frac{1}{h} \int_0^h \frac{\partial v}{\partial x} \, dz = \frac{1}{h} \left\{ \frac{\partial}{\partial x} \left[ \int_0^h v \, dz - \frac{\partial h}{\partial x} \right] \right\}. \]

from where:

\[ \frac{1}{h} \int_0^h \frac{\partial v}{\partial x} \, dz = \frac{\partial \bar{v}}{\partial x}. \]

The latest we generalize to the general situation as:

\[ \frac{1}{h} \int_z^{Z+h} \frac{\partial v_i}{\partial x_j} \, dx_3 = \frac{\partial \bar{v}_i}{\partial x_j}, \quad \frac{1}{h} \int_z^{Z+h} \frac{\partial v_j}{\partial x_i} \, dx_3 = \frac{\partial \bar{v}_j}{\partial x_i}. \]
From here, substituting the above expressions into (36), we arrive at:

\[
\bar{d}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right), \quad i, j = 1, 2.
\]

The component \( \bar{d}_{33} \) is obtained in a straightforward manner as follows:

\[
\bar{d}_{33} = \frac{1}{h} \int_{Z}^{Z+h} \frac{\partial v_3}{\partial x_3} dx_3 = \frac{1}{h} (v_3|_{Z+h} - v_3|_{Z}) \approx \frac{1}{h} \frac{\partial h}{\partial t}.
\]

Finally, the components \( \bar{d}_{i3} \) with \( i = 1, 2 \), can be calculated using the assumption that the flow is that of an infinite landslide:

\[
\bar{d}_{i3} = \frac{1}{h} \left\{ \int_{Z}^{Z+h} \frac{\partial v_i}{\partial x_3} dx_3 + \int_{Z}^{Z+h} \frac{\partial v_3}{\partial x_i} dx_3 \right\}.
\]

The second term in the above equality is zero, as \( v_3 \) does not depend on \( x_i \), and we arrive at:

\[
\bar{d}_{i3} = \frac{1}{h} \int_{Z}^{Z+h} \frac{\partial v_i}{\partial x_3} dx_3.
\]

This term is usually evaluated assuming the flow structure is that of an infinite landslide, which depends on the rheological model being chosen.

4.3. Bagnold Fluids

4.3.1. Introductory remarks

We shall assume that the soil is saturated, has a porosity \( n \) and consists of particles having a density \( \rho_s \).

The components of the stress tensor for a Bagnold fluid were given in (26)–(28). We shall write them for completeness below:

\[
\sigma_{13} = a_i \sin \alpha_i \rho \lambda^2 c^2 \left( \frac{dv_1}{dx_3} \right)^2,
\]

\[
\sigma_{33} = p + a_i \cos \alpha_i \rho \lambda^2 c^2 \left( \frac{dv_1}{dx_3} \right)^2,
\]

where we have assumed compression stresses are positive. In order to simplify
the notation, we shall write the stresses as:

\[ \tau = \mu B \sin \phi B \left( \frac{\partial \nu}{\partial z} \right)^2, \]

\[ (40a) \]

\[ \sigma_v = p + \mu B \cos \phi B \left( \frac{\partial \nu}{\partial z} \right)^2, \]

\[ \sigma_h = \sigma_v, \]

where we have introduced the material parameter

\[ (40b) \]

\[ \mu_B = a_i \rho \lambda^2 c_i^2 \]

and denoted \( \phi_B = \alpha_i \).

There are some remarks concerning the stress state. First of all, the model was proposed by Bagnold who performed experiments on grains with the same density as water, in order to avoid the effects of a variation of vertical and horizontal stresses due to gravity. When the mixture is at rest, the stress tensor is hydrostatic, all the components \( \sigma_{ii} \) of the total stress tensor being equal to \( p \). If we consider a granular mixture which is starting to move, this stress state is not valid unless the soil has liquefied, i.e., the mean effective confining pressure is zero.

The total vertical stress in the soil phase is given by:

\[ (41) \]

\[ \sigma_v / g (h - z) = (1 - n) \rho'_s = \rho'_d, \]

where the submerged density of soil particles \( \rho'_s \) is

\[ (42a) \]

\[ \rho'_s = \rho_s - \rho_w, \]

and

\[ (42b) \]

\[ \rho'_d = (1 - n) \rho'_s. \]

If the soil has liquefied, the increment in pore pressure should be equal to \( \sigma_v \). The total pore pressure will be:

\[ (43) \]

\[ p_w / g (h - z) = \rho_w + (1 - n) \rho'_s \]

\[ = \rho_w + (1 - n) (\rho_s - \rho_w) \]

\[ = \rho. \]
The total pressure acting on the basal surface is obtained by adding the contributions of solid and fluid phases. Therefore, the pressure term is given by:

\[ p = g(h - z) \{ (1 - n) \rho_s + n \rho \} . \]

If we plot the stress path in the \((\sigma, \tau)\) plane when the rotation speed in the rheometer is increasing, we obtain the results shown in Fig. 8. Starting from the point O, the effective stress state \((p', \tau)\) moves along a straight line with slope \(\phi_B\). The Mohr circle lies outside this line, and indeed it is tangent to a straight line having a slope \( \tan \phi_{dyn} \) given by:

\[ \sin \phi_{dyn} = \tan \phi_B . \]

So far, we have avoided proposing any relation between the residual angle of friction of the granular material we are considering and the angles \(\phi_B\) and \(\phi_{dyn}\). Indeed, the former can be higher. In this case, the stress state would be outside the Mohr-Coulomb failure surface, which is consistent with viscoplasticity, where stresses can be outside the yield surface. We should keep in mind that Bagnold measured stresses once equilibrium has been reached, and it is possible that stress states of higher intensity exist.

### 4.3.2. Flow structure and basal friction

The vertical distribution of shear stress was obtained in (33):

\[ \tau = \tau_B \left( 1 - \frac{z}{h} \right) = \tau_B (1 - \xi) , \]

where the shear stress is given by (40)

\[ \tau = \mu_B \sin \phi_B \left( \frac{\partial v}{\partial z} \right)^2 . \]
From the above expressions after integration, and using the boundary condition \( v(0) = 0 \), we derive:

\[
\begin{align*}
\Rightarrow v &= \frac{2h}{3} \sqrt{\frac{\tau_b}{\mu B \sin \phi_B}} \left\{ 1 - (1 - \xi)^{\frac{3}{2}} \right\}.
\end{align*}
\]

If we denote the maximum velocity, which takes place at the surface, by \( v_P \) then:

\[
\begin{align*}
v &= v_P \left\{ 1 - (1 - \xi)^{\frac{3}{2}} \right\}
\end{align*}
\]

with

\[
\begin{align*}
v_P &= \frac{2h}{3} \sqrt{\frac{\tau_b}{\mu B \sin \phi_B}}.
\end{align*}
\]

The averaged velocity \( \bar{v} \) is obtained by integrating along the vertical direction the velocity, and it reads:

\[
\begin{align*}
\bar{v} &= \frac{3}{5} v_P = \frac{2h}{5} \sqrt{\frac{\tau_b}{\mu B \sin \phi_B}}.
\end{align*}
\]

From here, it is immediate to obtain the basal friction as:

\[
\begin{align*}
\tau_b &= \frac{25}{4} \mu B \sin \phi_B \frac{\bar{v}^2}{h^2}.
\end{align*}
\]

**4.3.3. Momentum correction factor**

The correction factor in the depth averaged momentum equation will be obtained using its definition:

\[
\alpha = \frac{1}{h} \frac{1}{\bar{v}^2} \int_0^1 (v - \bar{v})^2 h d\xi = \frac{\int_0^1 v^2 d\xi}{\bar{v}^2} - 1.
\]

In our case, we have:

\[
\int_0^1 v^2 d\xi = \frac{9}{20} h v_P^2,
\]

from where the momentum correction factor is derived to be \( \alpha = 0.25 \).

**4.3.4. Depth averaged stresses**

The stress tensor was found to be given by (28):

\[
\sigma = -p I + \left\{ 4a_i \sin \alpha_i \rho \lambda^2 c^2 \left( I_{2D} \right)^{\frac{1}{2}} \right\} \text{dev} (d) - \left\{ 4a_i \cos \alpha_i \rho \lambda^2 c^2 \right\} d^2
\]
or, with the notation introduced in this subsection:

\[
\sigma = -pI + \left\{4\mu_B \sin \phi_B \left( I_{2D} \right)^{\frac{1}{2}} \right\} \text{dev}(d) - \left\{4\mu_B \cos \phi_B \right\} d^2.
\]

Please note that according to that, as explained above, \( \bar{p} = g(h - z)(\rho + n\rho_d') \), where \( g(h - z)n\rho_d' \) is the excess pore pressure which is causing liquefaction.

The depth averaged stress tensor will be assumed to be:

\[
\bar{\sigma} = -\bar{p}I + \left\{4\mu_B \sin \phi_B \left( \bar{I}_{2d} \right)^{\frac{1}{2}} \right\} \text{dev}(\bar{d}) - \left\{4\mu_B \cos \phi_B \right\} \bar{d}^2.
\]

We shall consider first the linear term \( 4\mu_B \sin \phi_B \left( \bar{I}_{2d} \right)^{\frac{1}{2}} \) dev(\( \bar{d} \)). According to (37), the components of the averaged rate of deformation tensor can be obtained from the averaged velocities:

\[
\bar{d}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) , \quad i, j = 1, 2.
\]

In order to evaluate \( \bar{I}_{2d} \) we shall need the components involving the vertical direction \( \bar{d}_{13}, \bar{d}_{23} \) and \( \bar{d}_{33} \). The latter was proposed to be approximated by (38):

\[
\bar{d}_{33} = \frac{1}{h} \frac{\partial h}{\partial t}.
\]

Concerning \( \bar{d}_{13}, \bar{d}_{23} \) they are approximated as:

\[
\bar{d}_{k3} = \frac{1}{h} \int_0^h \frac{\partial v_k}{\partial x_3} dz , \quad k = 1, 2.
\]

In the case of the infinite landslide, the above integral is taken easily resulting in:

\[
\bar{d}_{k3} = \frac{5}{3} \bar{v}_k , \quad k = 1, 2.
\]

Next, we shall obtain the quadratic term \( - \left\{4\mu_B \cos \phi_B \right\} \bar{d}^2 \). Note that \( \bar{d}^2 \) is the average of \( d^2 \) and not the second power of the averaged tensor.

We shall use an auxiliary reference system \((x, y, z)\), where \( z \equiv x_3 \), \( x \) is horizontal and parallel to the averaged horizontal velocity \( \bar{v} \), and \( y \) orthogonal to \( x \) and \( z \). We shall further assume that the flow structure is that of a simple
shear flow (infinite landslide). In this reference system, the $d_2^2$ only has two non zero components, $d_{xx}^2 = d_{zz}^2$, which are given by:

\begin{equation}
\frac{d_{xx}^2}{h} = \frac{1}{h} \int_0^h \left( \frac{\partial v}{\partial z} \right)^2 dz.
\end{equation}

The integral can be easily computed, using (33) and (47):

\[ \tau = \tau_b \left( 1 - \frac{z}{h} \right) = \tau_b \left( 1 - \xi \right), \]

from where:

\begin{equation}
\left( \frac{\partial v}{\partial z} \right)^2 = \frac{\tau_b}{\mu_B \sin \phi_B} \left( 1 - \xi \right).
\end{equation}

Substituting into (55), and integrating, we obtain:

\begin{equation}
\frac{d_{xx}^2}{h} = \frac{\tau_b}{2\mu_B \sin \phi_B}.
\end{equation}

This expression can be written in a more convenient way taking into account (51) which relates the basal shear to the averaged velocity. The result is:

\begin{equation}
\frac{d_{xx}^2}{h} = \frac{25}{8} \bar{v}^2 h^2.
\end{equation}

From where we obtain the contribution of the quadratic term to the stress tensor as:

\begin{equation}
\sigma_{xx}^{\text{dyn}} = \frac{25}{8} \bar{v}^2 {h^2} \mu_B \cos \phi_B.
\end{equation}

This component has to be changed back to the reference system $(x_1, x_2, x_3)$ and added to components in axes $i, j$

\[ \frac{\partial}{\partial x_j} \left( h \sigma_{ij} \right). \]

4.3.5. Generalization

If we use an exponent $m$ different from 2 in Bagnold model that is the model reads:

\begin{equation}
\tau = \mu_B \sin \phi_B \left( \frac{\partial v}{\partial z} \right)^m,
\end{equation}

\begin{equation}
\sigma_v = p + \mu_B \cos \phi_B \left( \frac{\partial v}{\partial z} \right)^m,
\end{equation}

\[ \sigma_h = \sigma_v. \]
In this case we obtain:
(i) The velocity profile is:

\[ v = v_P \left\{ 1 - \left( 1 - \frac{m+1}{m} \right) \right\}, \]

where

\[ v_P = \frac{m}{m + 1} h \left( \frac{\tau_b}{\mu_B \sin \phi_B} \right)^{\frac{1}{m}}. \]

(ii) The averaged velocity is:

\[ \bar{v} = \frac{m + 1}{2m + 1} v_P. \]

(iii) The bottom friction is:

\[ \tau_b = \left( \frac{2m + 1}{m} \right)^{m} \frac{\mu_B \bar{v}^m}{h^m}. \]

(iv) The momentum correction factor is given by:

\[ \alpha = \frac{m}{3m + 2}. \]

This relation is depicted in Fig. 9. The value of the correction factor increases with the exponent \( m \), reaching a maximum asymptotic value of 1/3.

(v) The dynamic contribution to the averaged stress along the direction of the flow is:

\[ \sigma_{xx}^{dyn} = \frac{1}{2} \left( \frac{2m + 1}{m} \right)^{m} \mu_B \cos \phi_B \bar{v}^m h^m. \]

5. The Critical State in flowing geomaterials

In Classical Soil Mechanics it is usually assumed that residual conditions—defined as a state where the soil is sheared at constant effective stress—take place at a line in the 3D space \((e, p', q)\) referred to as Critical State Line (CSL) (Parry, [13]).

The projections onto the planes \((e, \ln p')\) and \((p', q)\) are the straight lines:

(i) \( e = \Gamma^* - \lambda \ln p' \),

(ii) \( q = M p' \).
These lines are depicted in Fig. 10.

We have plotted two undrained paths corresponding to a normally consolidated clay (path AB) and a overconsolidated clay (path CB) with equal initial voids ratio at points A and C. Both paths end at the same point of the CSL.

We shall consider next experimental observations on constant volume and constant pressure rheometers. The tests are run at a constant strain rate, and it is observed that at steady state the pressure and voids ratio depend on it. Therefore, there is not a unique CSL in dynamics. If volume is kept constant (constant voids ratio), an increase of rate of strain causes an increase in
pressure. In constant pressure rheometers, an increase of the rate of deformation causes an increase of voids ratio. These tests have been sketched in Fig. 11. Concerning the CSL in the \((p', q)\) plane, Bagnold [1] observed a constant slope for the range of values of rate of strain considered (dynamic equilibrium conditions).

Additional experimental effort is still needed to clarify these aspects.

6. Conclusions

This paper has presented constitutive and rheological models which can be applied to describe the behaviour of fluidized geomaterials both at the initiation and during the propagation phase. Concerning the former, generalized plasticity has been shown to provide a framework within which liquefaction and collapse can be modelled. Once the soil has become a fluid, we propose the use of rheological models to describe its behaviour. Here we describe in detail Bagnold’s model, which in our opinion can be applied to liquefied geomaterials.

REFERENCES


