CANTILEVER BEAM NATURAL FREQUENCIES IN CENTRIFUGAL INERTIA FIELD

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ABSTRACT: In the advanced mechanical science the well known fact is that the gravity influences on the natural frequencies and modes even for the vertical structures and pillars. But, the condition that should be fulfilled in order for the gravity to be taken into account is connected with the ration between the gravity value and the geometrical cross section inertia. The gravity is related to the earth acceleration but for moving structures there exist many other acceleration exaggerated forces and such are forces caused by the centrifugal accelerations. Large rotating structures, as wind power generators, chopper wings, large antennas and radars, unfolding space structures and many others are such examples. It is expected, that acceleration based forces influence on the structure modal and frequency properties, which is a subject of the present investigations.

In the paper, rotating beams are subject to investigations and modal and frequency analysis is carried out. Analytical dependences for the natural resonances are derived and their dependences on the angular velocity and centrifugal accelerations are derived. Several examples of large rotating beams with different orientations of the rotating shaft are presented. Numerical experiments are conducted. Time histories of the beam tip deflections, that depict the beam oscillations are presented.

KEY WORDS: Natural frequencies, systems with distributed parameters, differential transformation method (DTM), numerical methods.

1. INTRODUCTION

Rotating machines are the basic units in many structures and industrial applications. Most of them are on-board machines that cause inertial excited vibrations and non-stationary phenomena. The rotor behaviour could be also influenced by the base movement, that leads to undesirable vibrations and is one of the main reasons for non-stationary phenomena. As it is pointed out in [1], the gravity could also cause influence on the magnitude of the modal frequencies. In many papers, the natural

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frequencies are estimated on the basis of the modal analysis, taking no into account many other effects and phenomena. For example, the gravity, induced inertia forces and many others.

Numerical methods for dynamics simulation of rigid and flexible system propose many advantages and possibilities for analysis of complex systems with many degrees of freedom and nonlinear effects and are successfully applied in the design process. But, the outcomes of the computations are only to visualize the results and to convince the engineers that they are in the right direction, but not to proof their conclusions. Although some restrictions of their application, the analytical methods could lead to fundamental conclusions and proofs. The analytical functions derived could clearly show the influence of the design and the motion parameters on the system behaviour.

The present paper analyses the influence of the centrifugal inertia on the natural frequencies of high speed rotating beam. Examples of such applications are the propellers of the wind generators, helicopters and others. Analytical dependences are derived for estimation of the natural frequencies as a function of the beam angular velocity. Numerical examples present the dependences and the change of the vibration oscillations as a result of the angular velocity.

2. NOMENCLATURE

- $m$ – distributed mass [kg/m];
- $J$ – geometrical inertia moment [m$^4$];
- $E$ – modulus of the elasticity [MN/m$^2$];
- $\omega$ – angular velocity [s$^{-1}$];
- $p_i$ – $i$-th natural frequency [s$^{-1}$];
- $l$ – length [m];
- $N(x)$ – inertia force along the axis, perpendicular to the beam cross section [N];
- $x$ – length coordinate of a point of the middle axis of the beam $0 \leq x \leq l$;
- $\Phi$ – non-dimensional inertia force [-];
- $y(x, t)$ – transversal deflection along the beam longitudinal axis [m];
- $u(x)$ – amplitude function.

3. THEORETICAL CONSIDERATIONS

The partial differential equation, describing the free vibrations of a conservative system, for example a beam with a constant cross-section in a centrifugal inertia field (Fig. 1) could be presented as follows:

\[
\frac{\partial^2}{\partial x^2} \left[ EJ \frac{\partial^2 y}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ N(x) \frac{\partial y}{\partial x} \right] + m \frac{\partial^2 y}{\partial t^2} = 0.
\]
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\( Z \) - angular velocity \( [s^{-1}] \);

\( i_p \) - \( i \)-th natural frequency \([s^{-1}]\);

\( l \pm \) length \([m]\);

\( x_{N} \) - inertia force along the axis, perpendicular to the beam cross-section \([N]\);

\( x \pm \) length coordinate of a point of the middle axis of the beam \([\text{m}]\);

\( l x \) \( d_d \) \( 0 \) - non-dimensional inertia force \([-\]);

\( t x \ y m \), \( x u \) - amplitude function;

3. Theoretical considerations

The partial differential equation, describing the free vibrations of a conservative system, for example a beam with a constant cross-section in a centrifugal inertia field (Fig. 1) could be presented as follows:

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} - m^2 \omega^2 \left( l^2 - x^2 \right) \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = 0.
\end{align*}
\]

The inertia force acting in the beam cross-section on a distance \( x \) from the coordinate system origin is presented by

\[
N(x) = m(l - x)r \omega^2 = \frac{m}{2} \omega^2 (l^2 - x^2),
\]

where

\[
r = x + \frac{l - x}{2} = \frac{1}{2} (l + x).
\]

After substitution of (2) in (1) one obtains:

\[
\frac{\partial^2}{\partial x^2} \left[ EJ \frac{\partial^2 u(x)}{\partial x^2} \right] - \frac{m}{2} \omega^2 (l^2 - x^2) \frac{\partial^2 u(x)}{\partial x^2} + \frac{m}{2} \omega^2 x \frac{\partial u(x)}{\partial x} + m \frac{\partial^2 u(x)}{\partial t^2} = 0.
\]

In the conjunction with the procedures valid for the linear partial differential equations [1], [2], [3], one looks for the solution of (3) in the following form:

\[
y(x, t) = u(x) \cos (pt).
\]

After substitution of the corresponding derivatives of (4) in (3), one obtains

\[
\frac{\partial^2}{\partial x^2} \left[ EJ \frac{\partial^2 u(x)}{\partial x^2} \right] - \frac{m}{2} \omega^2 (l^2 - x^2) \frac{\partial^2 u(x)}{\partial x^2} + \frac{m}{2} \omega^2 x \frac{\partial u(x)}{\partial x} + mp^2 u(x) = 0.
\]
To obtain general solutions of Eq. (5), it is necessary to present it in dimensionless form by means the following substitution:

$$\eta = \frac{x}{l} ; \quad \Phi = \omega^2 \frac{ml^4}{EJ} ; \quad p = \frac{\omega^2}{ml^4} \sqrt{\frac{EJ}{ml^4}} ; \quad u(x) = l\phi(\eta) ,$$

where the Eq. (5) is transformed in

$$\frac{\partial^2}{\partial \eta^2} \left[ a\frac{\partial^2 \phi(\eta)}{\partial \eta^2} \right] - \frac{\Phi}{2} A \frac{\partial^2 \phi(\eta)}{\partial \eta^2} + \frac{\Phi}{2} B \frac{\partial \phi(\eta)}{\partial \eta} - b \Phi^3 \phi(\eta) = 0 ,$$

where $a = 1$, $A = 1 - \eta^2$, $B = 2\eta$, $b = 1$.

According to the differential transformation method (DTM) [2,5,6], the solution of the homogeneous differential equation (6) is presented in the following form:

$$(7) \quad \phi(\eta) = f(0) + f(1) \eta + f(2) \eta^2 + f(3) \eta^3 + \cdots ,$$

where

$$\phi(\eta) = \sum_{k=0}^{N} f(k)\eta^k$$

is the image of

$$u(x) = \sum_{k=0}^{N} f(k)x^k .$$

Actually the function $f(k)$ is the differential transformation of/or image of $u(x)$. After the dimensionless transformation of Eq. (5) the coordinate function, that is to be obtained, using the following dependence:

$$(8) \quad f(k) = \frac{1}{k!} \left[ \frac{d^k \phi(\eta)}{d\eta^k} \right]_{\eta=0} ,$$

which determines the coefficients in the Taylor’s series, Eq. (7). This approach is similar to the method of T-transformation [3,4]. If $a^*(k)$, $A^*(k)$, $B^*(k)$ and $b^*(k)$
are the images of $a(\eta)$, $A(\eta)$, $B(\eta)$ and $b(\eta)$, respectively and presented in Table 1, so the differential equation (6) is transformed in the following system of algebraic equations:

\[
\left\{ \Phi \frac{1}{2} \left[ -A^* (2) (k - 1) k + B (1) k - b^* (0) \right] f (k) \right\} - \Phi \frac{1}{2} A^* (0) (k + 1) (k + 2) f (k + 2) + a^* (0) (k + 1) (k + 2) (k + 3) (k + 4) f (k + 4) = 0 .
\]

The boundary conditions for estimation of the natural frequencies and modes of a cantilever beam are as follows [6], [7] and [9]:

\[
\begin{align*}
y (0, t) & \to u (0) \to f (0) \equiv 0 , \\
\frac{\partial y (0, t)}{\partial x} & \to \frac{\partial u (0)}{\partial x} \to f (1) \equiv 0 , \\
\frac{\partial^2 y (l, t)}{\partial x^2} & \to \frac{\partial^2 u (l)}{\partial x^2} \to \sum_{k=2}^{N} k (k - 1) f (k) \equiv 0 , \\
\frac{\partial^3 y (l, t)}{\partial x^3} & \to \frac{\partial^3 u (l)}{\partial x^3} \to \sum_{k=3}^{N} k (k - 1) (k - 2) f (k) \equiv 0 .
\end{align*}
\]

So, these four equations, together with $(N - 4)$ equations (9), are algebraic system of $(N)$ equations with more than $N$ unknowns. A necessary and a sufficient condition such systems to have a real solution, it is the major determinant to be equal zero. This is actually the frequency equation, from which the natural frequency of the rotating beam could be computed. The determinant is a high level polynomial with respect to $\wp^2$ (depends on the desirable precision, i.e., form the number $N$). From the solution of this polynomial, one computes the natural frequencies by the following relation:

\[
p_i = \frac{\wp_i^2}{I^2} \sqrt{\frac{EJ}{m}} .
\]

The precision of the approach proposed depends on the number of members in the Taylor’s series. Relatively, precise values for the first two natural frequencies for bodies with constant mass characteristics could be achieved for $18 \leq N \leq 22$. In engineering, it is admissible computation of the first $N_m$ modes of vibration, which adequately simulate the dynamic behaviour of the structure.
4. Numerical Example

Input data for the cantilever beam: diameter internal 0.26 [m], external 0.3 [m]; geometrical inertia moment \( J = 1.734.10^{-4} [m^4] \); distributed mass \( m = 138 \) [kg/m]; modulus of the elasticity \( E = 2.1.10^{11} [N/m^2] \); \( p_1 = 18.01 [s^{-1}] \); \( p_2 = 113.25 [s^{-1}] \) for \( \omega = 0 \). The computations in relation to the presented methodology in [4], [5], and application of the method (DTM), are presented in Table 2 for three sizes of the square matrices. The values of \( N \) are 19, 30 and 46 and the values of \( \varphi_i \) are shown in Table 2. It could be seen, that for the first two natural frequencies it is enough the value of \( N \) to be within the range of \( 15 \leq N \leq 20 \) which, actually, is the dimension of the determinant. For the frequencies in the range \( 3 \leq p_i \leq 6 \) the dimension is to be in the range of \( 30 \leq N \leq 50 \) [8].

<table>
<thead>
<tr>
<th>( N = 19 )</th>
<th>( N = 30 )</th>
<th>( N = 46 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_1^\Phi )</td>
<td>17.1171</td>
<td>17.1174</td>
</tr>
<tr>
<td>( \varphi_2^\Phi )</td>
<td>511.4238</td>
<td>511.4277</td>
</tr>
<tr>
<td>( \varphi_3^\Phi )</td>
<td>4219.6769</td>
<td>3907.9905</td>
</tr>
<tr>
<td>( \varphi_4^\Phi )</td>
<td>5451.5228</td>
<td>22461.2884</td>
</tr>
</tbody>
</table>

5. Numerical Verification of the Phenomenon

The effect of the phenomenon, discussed in the previous sections and investigated analytically will be demonstrated for another example of dynamic simulation of the rotational beam of Kane [10], Fig. 2, widely discussed in the scientific literature. The material and the mass and the inertia characteristics of the beam are: modulus of elasticity \( E = 7 \times 10^{10} [N/m^2] \); shear modulus \( G = 2.9 \times 10^{10} [N/m^2] \); length \( L = 10 [m] \); density \( \rho = 3000 [kg/m^3] \); cross section area \( S = 4 \times 10^{-4} [m^2] \); cross-section moments of inertia \( J_x = J_z = \frac{1}{2} J_c = 2 \times 10^{-7} [m^4] \). Static deflection,

![Fig. 2. The rotational beam of Kane [10].](image-url)
due to gravity is taken into account, where its value is −1.06 [m] in the beam initial position.

To demonstrate the influence of the centrifugal inertia forces, the beam starts to rotate with a constant acceleration till the desired angular velocity. Than the acceleration becomes equal to zero and the beam starts to rotate with constant angular velocity. From this moment on the angular velocity is constant and only centrifugal forces load the beam.

Two examples of the beam rotations with constant angular velocities of \( \omega = 6 \, [\text{s}^{-1}] \) and \( \omega = 1.5 \, [\text{s}^{-1}] \) are presented in Figs. 3 and 4, respectively. On each example, two cases are discussed – the amplitudes in the horizontal plane (without gravity impact) and in the vertical plane taking into account the gravity.

From the examples it could be easily concluded, that the angular velocity influence on the vibration frequency of the beam. The influence is much more visible if the gravity is taken into account. It must be pointed out that these vibrations do not coincide with the natural frequencies of the beam, but only show that the centrifugal forces change the frequency domain.

![Fig. 3. Time history of the amplitudes taking into account the gravity.](image)

![Fig. 4. Time history of the amplitudes without the effect of the gravity.](image)
6. CONCLUSIONS

The advantages of the method proposed for estimation of the modes and natural frequencies of bodies with distribute parameters insist in the following:

— solution of linear partial differential equations of arbitrary kind;
— precise solution for the first natural frequencies with small number algebraic equations;
— application of the algorithm for bodies with arbitrary size and shape of the cross-section, as well as of the physical characteristics along the longitudinal axis of the beam.

The major disadvantage is the necessity for computation of the determinants of high order, especially for beams with variable geometry, for which the matrix of the coefficients of the frequency equation is with many nonzero elements. The results obtained for the natural frequencies, especially for the first one, are approximated equally, that proofs their reliability. For a beam in centrifugal inertia field, the stiffness of the beam is confirmed, i.e., increasing the values of the natural frequencies as a result of the inertia forces along the beam longitudinal axis.

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