

NONLINEAR ANALYSIS OF A FUNCTIONALLY GRADED BEAM RESTING ON THE ELASTIC NONLINEAR FOUNDATION*

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ABSTRACT. This paper evaluates the nonlinear responses of a functionally graded (FG) beam resting on a nonlinear foundation. After derivation of fundamental nonlinear differential equation using the Euler-Bernouli beam theory, a semi analytical method has been used to study the response of the problem. The responses can be evaluated for both linear and nonlinear isotropic and FG beams individually. Adomians Decomposition and successive approximation methods have been used for solution of nonlinear differential equation. As numerical investigation, the beams with simply supported ends and linear and nonlinear foundations have been analyzed using this method.

KEY WORDS: Functionally graded beam, nonlinear response, nonlinear foundation, Adomians Decomposition Method (ADM).

1. Introduction

Beams are one of the most applicable structures in the scope of mechanical engineering and analysis of the structures. There are two main theories to analyze the beam under specific loads. Euler Bernoulli and Timoshenko theories are two mentioned. The first is applicable for beam with short width and consequently only bending deformation is considered. In contrast, second theory is applicable for wide beam. In this paper, linear and nonlinear responses of a beam resting on the linear and nonlinear foundations with constant and variable properties can be evaluated analytically. A brief review on the literature indicates that there is no published research containing functionally graded beam resting on the linear and nonlinear foundations.

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Pradhan and Murmu [1] presented thermo-mechanical vibration analysis of functionally graded (FG) beams and functionally graded sandwich (FGSW) beams. Both beams have been assumed to be resting on two various foundations. Functionalities have been considered along the thickness direction. The effect of different parameters such as temperature distribution, power-law index and parameters of foundation has been considered on the vibration characteristics of the beam. Ying, Lu and Chen [2] investigated on the bending and free vibration of functionally graded beams resting on a Winkler–Pasternak elastic foundation. They used two-dimensional theory of elasticity for simulation of deformations. Exponentially function has been used for variation of material properties along the thickness direction.

Banerjee et al [3] studied the nonlinear deformation of beam made of isotropic material. They employed analytical and numerical methods for analysis of beam. Analytical method was performed by Adomians Decomposition Method (ADM) and shooting method was employed as a numerical method. Kapuria et al [4] presented vibration analysis of a functionally graded beam. They regarded the normal and shear stresses in that study.

A new beam theory has been developed by Sina, Navazi and Haddadpour [5] in order to analyze free vibration of functionally graded beams. The beam properties are assumed to be varied through the thickness following a simple power law distribution in terms of volume fraction of material constituents. Xiang and Yang [6] investigated on the free and forced vibrations of a laminated functionally graded beam of variable thickness under thermal loads. The beam was manufactured of a homogeneous substrate and two non-homogeneous functionally graded layers. A two dimensional analysis has been used and therefore, both the axial and rotary inertia of the beam were considered in that analysis. Vibrations of axially moving flexible beams made of functionally graded materials were studied by Piovan and R. Sampaio [7]. The used model was a thin-walled beam with annular cross-section.

Li [8] presented a new approach for analyzing the static and dynamic behaviour of functionally graded beams (FGB) with the rotary inertia and shear deformation included. All material properties were arbitrary functions along the beam thickness. A single fourth-order governing partial differential equation was derived and all physical quantities were expressed in terms of the solution of the resulting equation. As a case study, the Euler–Bernoulli and Rayleigh beam theories have been derived by reducing the Timoshenko beam theory. Benatta et al [9] used high-order flexural theories for short functionally graded symmetric beams under three-point bending. The governing equations were obtained using the principle of virtual work (PVW). Kadoli, Akhtar and

Ganesan [10] presented static behaviour of functionally graded metal–ceramic (FGM) beams under ambient temperature by using a higher order shear deformation theory. The finite element form of static equilibrium equation for FGM beam was presented using the principle of stationary potential energy. The effect of power law exponent for various combination of metal–ceramic FGM beam on deflection and stresses were investigated. Yu, Yuan-Yuan and Chang [11] developed a nonlinear mathematical model for large deformation analysis of beams with discontinuity conditions and initial displacements. The differential quadrature element method (DQEM) was applied to discretize the nonlinear mathematical model. A rectangular and simply supported functionally graded beam with thick thickness under transverse loading has been investigated by Ben-Oumrane et al [12]. First and higher order shear deformation theories have been used for analysis. They assumed that Young’s modulus vary continuously throughout the thickness direction according to the volume fraction of constituents.

The out-of-plane free vibration analysis of thin and thick functionally graded circular curved beams on two-parameter elastic foundation was presented by Malekzadeh et al [13]. They used first-order shear deformation theory (FSDT) in order to account the effects of shear deformation and rotary inertia due to both torsion and flexural vibrations. The material properties were assumed to be graded in radial direction of the beam curvature. Yousefi and Rastgoo [14] analysed free vibration of functionally graded spatial curved beams.

Based on the reported research, one can find that there is no study to addresses the nonlinear analysis of a beam resting on the linear and nonlinear foundations manufactured of materials with variable properties along the longitudinal direction. Furthermore, for solution of derived system of nonlinear differential equation, a semi analytical approach has been presented.

2. Formulation for isotropic beam with constant thickness

2.1. Nonlinear foundation

This section presents the nonlinear differential equation of an isotropic beam subjected to nonlinear foundation. From strength of material, we have the relation between deflection and bending momentum as follows [3]:

$$(1) \quad M = -EI \frac{d^2 y}{dx^2},$$

where M is bending momentum along the outward axis and y is deflection of the beam. EI is rigidity modulus. Two times differentiating with respect to x

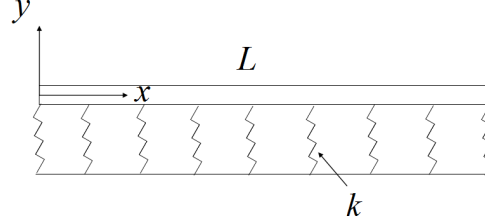


Fig. 1. The schematic figure of a beam resting on the foundation

presents the load-deflection equation as follows:

$$(2) \quad EI \frac{d^4 y}{dx^4} = -q(x).$$

where, q is load per unit length of beam. The positive direction of q is coincident with the positive direction of deflection.

It was assumed that foundation has nonlinear behaviour of order two. Therefore we have:

$$(3) \quad q = k_0 + k_1 y + k_2 y^2.$$

By substitution of above function into Eq. 2, we have:

$$(4) \quad EI \frac{d^4 y}{dx^4} = -(k_0 + k_1 y + k_2 y^2).$$

Above nonlinear differential equation can be solved using the Adomians Decomposition Method. This method decomposes differential equation to linear and nonlinear operators and proposes solution using a successive procedure [3, 16].

$$(5) \quad \begin{aligned} EI \frac{d^4 y}{dx^4} + k_1 y + k_2 y^2 &= -k_0, \\ L(y) + R(y) + N(y) &= g(x), \\ L &:= \frac{d^4}{dx^4}, \quad R := \frac{k_1(\dots)}{EI}, \quad N := \frac{k_2(\dots)^2}{EI}, \quad g := \frac{-k_0}{EI}, \end{aligned}$$

where, L is largest linear operator, R is other linear operators, N is nonlinear operator and g is a function that appears on the right side of the nonlinear

equation. After defining the linear and nonlinear operators in nonlinear differential equations, the solution procedure can be presented by multiplication of nonlinear equation in L^{-1} , as follows:

$$\begin{aligned}
 L^{-1} &:= \int \int \int \int (\dots) dx dx dx dx \\
 y_{n+1} &:= -L^{-1}(R(y_n)) - L^{-1}(N(y_n)) \quad n = 0, 1, 2, \dots \\
 (6) \quad y_0 &:= L^{-1}(g(x)) + c_0 + c_1x + c_2x^2 + c_3x^3 \\
 &= -\frac{k_0x^4}{24EI} + c_0 + c_1x + c_2x^2 + c_3x^3,
 \end{aligned}$$

where, the defined constants c_i , $i = 1, \dots, 4$ must be obtained by imposing the boundary conditions on the zero'th order solution y_0 .

The longitudinal distribution of nonlinear deflection of an isotropic beam with constant thickness in terms of different values of nonlinear index (k_2) is shown in Fig. 2. It can be concluded that the large values of nonlinear loading parameter (k_2) has an important effect on the responses of the beam.

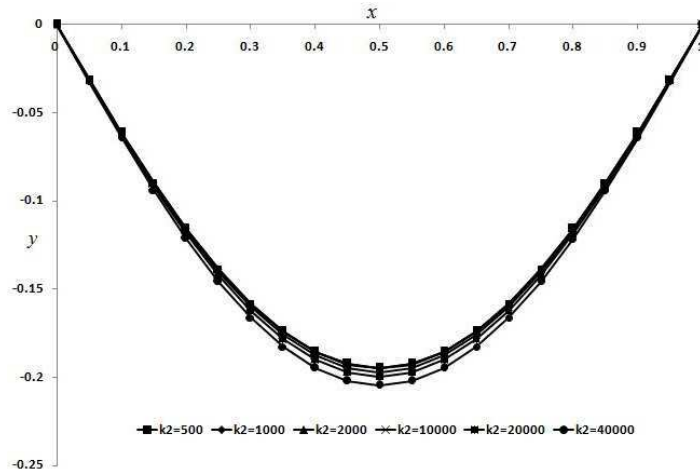


Fig. 2. Longitudinal distribution of nonlinear deflection of an isotropic beam with constant thickness in terms of different values of nonlinear index (k_2)

The percentage of difference between linear and nonlinear results for different values of nonlinear parameter can be presented in Fig. 3. The obtained results indicate that the percentage of difference between linear and nonlinear results increases with increasing the nonlinear index (k_2).

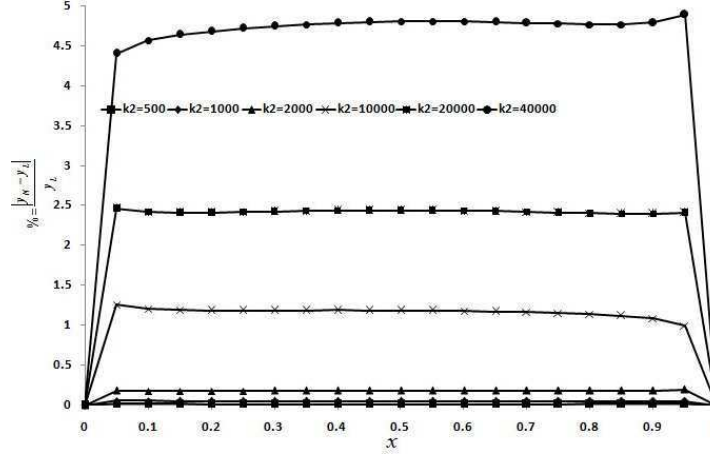


Fig. 3 The percentage of difference between linear and nonlinear results for different values of nonlinear parameter

3. Formulation for FG beam with variable thickness

The nonlinear formulation for a functionally graded beam with variable thickness and subjected to nonlinear foundation can be studied in the present section.

3.1. Linear foundation

It is assumed that beam is graded along the longitudinal direction. For the beam with mentioned condition, we have following relation:

$$\begin{aligned}
 M &= -EI(x) \frac{d^2 y}{dx^2} \\
 (7) \quad V &= \frac{dM}{dx} = -\frac{d}{dx} \left(EI(x) \frac{d^2 y}{dx^2} \right) \rightarrow q(x) = -\frac{d}{dx} \left(\frac{d}{dx} \left(EI(x) \frac{d^2 y}{dx^2} \right) \right) \rightarrow \\
 &\frac{d^2 [EI(x)]}{dx^2} \frac{d^2 y}{dx^2} + 2 \frac{d[EI(x)]}{dx} \frac{d^3 y}{dx^2} + EI(x) \frac{d^4 y}{dx^4} = -q(x).
 \end{aligned}$$

The solution can be decomposed into two linear and nonlinear responses in terms of foundation to be linear or nonlinear, respectively. For a linear foundation, we will have:

$$\begin{aligned}
 q(x) &= k_0 + k_1 y \rightarrow \\
 (8) \quad &\frac{d^2 [EI(x)]}{dx^2} \frac{d^2 y}{dx^2} + 2 \frac{d[EI(x)]}{dx} \frac{d^3 y}{dx^2} + EI(x) \frac{d^4 y}{dx^4} + k_1 y = -k_0.
 \end{aligned}$$

Adomians Decomposition Method can be applied for solution of above fourth order linear differential equation. For simplification of derived differential equation, all terms can be expressed in familiar form by division with $EI(x)$, as follows:

$$(9) \quad \frac{d^4 y}{dx^4} + \frac{(EI(x))'}{EI(x)} \frac{d^3 y}{dx^3} + \frac{2(EI(x))''}{EI(x)} \frac{d^2 y}{dx^2} + \frac{k_1}{EI(x)} y(x) = -\frac{k_0}{EI(x)} \rightarrow$$

$$L := \frac{d^4(\dots)}{dx^4}, \quad R := \frac{(EI(x))'}{EI(x)} \frac{d^3(\dots)}{dx^3} + \frac{2(EI(x))''}{EI(x)} \frac{d^2(\dots)}{dx^2} + \frac{k_1}{EI(x)} (\dots),$$

$$g := -\frac{k_0}{EI(x)}.$$

By defining the functionality of beam using an exponential function, we will have defined operators in the following form:

$$(10) \quad EI(x) = ae^{-bx}$$

$$L := \frac{d^4(\dots)}{dx^4}, \quad R := -b \frac{d^3(\dots)}{dx^3} + 2b^2 \frac{d^2(\dots)}{dx^2} + \frac{k_1}{a} e^{bx}(\dots), \quad g := -\frac{k_0}{a} e^{bx}.$$

Using the Adomians Decomposition Method, for zero'th order solution, we have:

$$(11) \quad y_0 = L^{-1}(g(x)) + c_0 + c_1 x + c_2 x^2 + c_3 x^3 =$$

$$\int \int \int \int \left(-\frac{k_0}{a} e^{bx}\right) dx dx dx dx + c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$y_0 = -\frac{k_0}{b^4 a} e^{bx} + c_0 + c_1 x + c_2 x^2 + c_3 x^3.$$

Higher order solutions can be obtained using successive approximation method, as follows:

$$y_{n+1} := -L^{-1}(R(y_n)) - L^{-1}(N(y_n)) \quad n = 0, 1, 2, \dots$$

$$y_{n+1} := -\frac{k_1}{EI} L^{-1}(y_n) + c_{n0} + c_{n1} x + c_{n2} x^2 + c_{n3} x^3$$

$$(12) \quad y_{n+1} := b \int \int \int \int \left(\frac{d^3(y_n)}{dx^3}\right) dx dx dx dx - 2b^2 \int \int \int \int \left(\frac{d^2(y_n)}{dx^2}\right) dx dx dx dx$$

$$-\frac{k_1}{a} \int \int \int \int (e^{bx} y_n) dx dx dx dx + c_{n0} + c_{n1} x + c_{n2} x^2 + c_{n3} x^3$$

$$y_{n+1} := b \int y_n dx - 2b^2 \int \int y_n dx dx$$

$$-\frac{k_1}{a} \int \int \int \int (e^{bx} y_n) dx dx dx dx + c_{n0} + c_{n1} x + c_{n2} x^2 + c_{n3} x^3$$

where, c_{n0} , c_{n1} , c_{n2} , c_{n3} are four constants of integration that appear in derived solution in n 'th step. These constants can be obtained by employing homogenized boundary conditions. Actual boundary conditions for a simply supported beam are:

$$(13) \quad x = 0, \quad L \rightarrow \begin{cases} y = 0 \\ M = 0 \end{cases}$$

Linear distribution of lateral deflection in terms of different values of non homogenous index of the problem is shown in Fig. 4. The obtained results indicate that the lateral deflection increases with increasing values of non homogenous index (b).

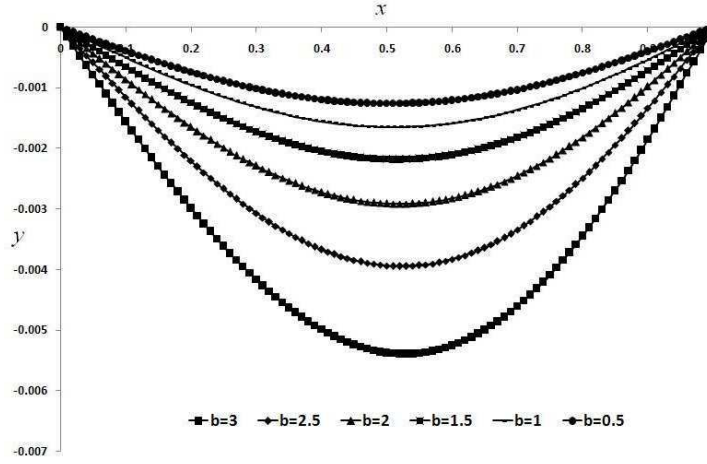


Fig. 4. Longitudinal distribution of linear lateral deflection in terms of different values of non homogenous index (b)

3.2. Nonlinear foundation

The effect of nonlinear foundation can be considered in this section. A cubic distribution has been considered for nonlinear distribution of elastic foundation. By employing this nonlinear foundation, we have nonlinear differential equation as follows:

$$(14) \quad q = (k_0 + k_1y + k_2y^2)$$

$$\frac{d^2[EI(x)]}{dx^2} \frac{d^2y}{dx^2} + 2 \frac{d[EI(x)]}{dx} \frac{d^3y}{dx^2} + EI(x) \frac{d^4y}{dx^4} + k_1y + k_2y^2 = -k_0.$$

The detail of Adomians Decomposition Method can be presented as follows:

$$\begin{aligned}
 & \frac{d^4 y}{dx^4} + \frac{(EI(x))''}{EI(x)} \frac{d^3 y}{dx^3} + \frac{2(EI(x))'}{EI(x)} \frac{d^2 y}{dx^2} + \frac{k_1}{EI(x)} y(x) + \frac{k_2}{EI(x)} y(x)^2 \\
 & = -\frac{k_0}{EI(x)} \rightarrow \\
 (15) \quad & L := \frac{d^4(\dots)}{dx^4}, R := \frac{(EI(x))''}{EI(x)} \frac{d^3(\dots)}{dx^3} + \frac{2(EI(x))'}{EI(x)} \frac{d^2(\dots)}{dx^2} + \frac{k_1}{EI(x)} (\dots) \\
 & N := \frac{k_2}{EI(x)} (\dots)^2, g := -\frac{k_0}{EI(x)}.
 \end{aligned}$$

From the above decomposition of nonlinear differential equation to basic linear and nonlinear operators and employing the mentioned method, we will have:

$$\begin{aligned}
 EI(x) &= ae^{-bx} \\
 L &:= \frac{d^4(\dots)}{dx^4}, R := b^2 \frac{d^3(\dots)}{dx^3} - 2b \frac{d^2(\dots)}{dx^2} + \frac{ke^{bx}}{a} (\dots), \\
 g &:= -\frac{k_0}{ae^{-bx}}, N := \frac{k_2 e^{-x}}{a} (\dots)^2,
 \end{aligned}$$

where, y_0 is zero'th order solution which can be obtained from $L^{-1}(g)$.

Longitudinal distribution of nonlinear lateral deflection is presented in Fig. 5. The effect of employed nonlinear analysis can be studied by comparison between linear and nonlinear responses. The percentage of difference between linear and nonlinear responses for different values of non homogenous index (b) is presented in Fig. 6.

The presented results in Fig. 6 indicate that the percentage of difference between linear and nonlinear responses increases with increasing the values of non homogenous index (b). This figure shows that the non homogenous index (b) has important effect on the difference of the linear and nonlinear responses.

4. Conclusion

The comprehensive analysis of different homogenous and non homogenous beam loaded under linear and nonlinear foundations have been performed in this paper. Adomians Decomposition Method has been employed for solution of obtained linear and nonlinear differential equations. The effect of non homogenous index and loading parameter has been considered on the linear and nonlinear system responses.

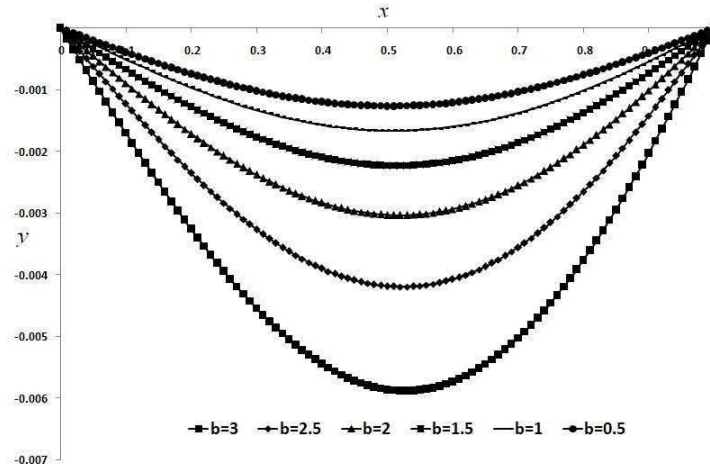


Fig. 5. Longitudinal distribution of nonlinear lateral deflection

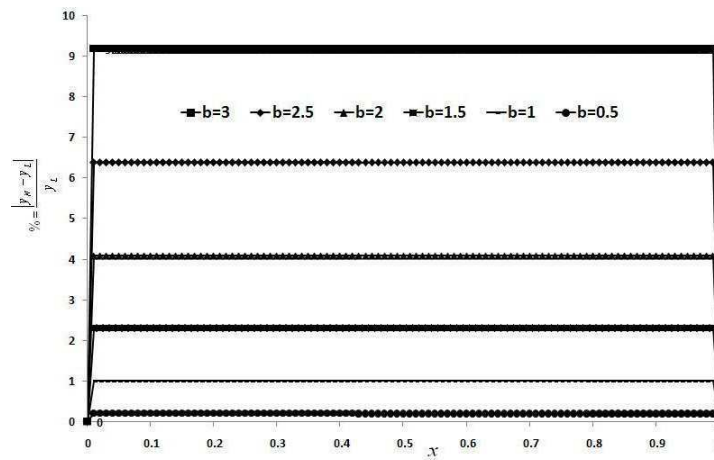


Fig. 6. The percentage of difference between linear and nonlinear responses

Investigation on the obtained results indicates that increasing the non homogenous index, increases both linear and nonlinear deflections. Furthermore, it can be concluded that increasing the non homogenous index, increases the percentage of difference between linear and nonlinear responses.

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