ANALYSIS OF MODE II CRACK IN BILAYERED COMPOSITE BEAM

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ABSTRACT. Mode II crack problem in cantilever bilayered composite beams is considered. Two configurations are analyzed. In the first configuration the crack arms have equal heights while in the second one the arms have different heights. The modulus of elasticity and the shear modulus of the beam un-cracked part in the former case and the moment of inertia in the latter are derived as functions of the two layers characteristics. The expressions for the strain energy release rate, $G$, are obtained on the basis of the simple beam theory according to the hypotheses of linear elastic fracture mechanics. The validity of these expressions is established by comparison with a known solution. Parametrical investigations for the influence of the moduli of elasticity ratio as well as the moments of inertia ratio on the strain energy release rate are also performed. The present article is a part of comprehensive investigation in Fracture mechanics of composite beams.

KEY WORDS: Crack, bilayered composite beam, strain energy release rate.

1. Introduction

Multilayered structural elements become common practice in the present-day constructions: the “sandwich” panels, the lattice steel beam and columns, compound timber beams, tailored composite beams, concrete beams reinforced by steel or composite layers [1, 2, 3, 4, 5].

In the present study the cantilever bilayered composite beam containing a single crack between the layers is investigated. The beam is manufactured by unidirectional fiber reinforced composites. It should be noted that there are no residual stresses and strains after the beam manufacturing.

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The beam is loaded by a concentrated force at the free end. Thus, the crack propagates in pure mode II. It should be mentioned that the development of beam specimens for experimental investigation of cracks is one of the main tasks in Fracture mechanics of composite beams. In this relation, analytical solutions are needed for the determination of parameters describing the crack behaviour [6, 7, 8]. Mode II crack behaviour has been explored in many works where the cantilever beam specimen is made of one composite material and, besides, the crack arms have the same heights [9, 10, 11].

Here, two configurations of the bilayered composite beam are considered: in the first one the crack arms are made of two different unidirectional fiber reinforced composites while in the second one the moments of inertia of the crack arms are different (Fig. 1).

![Fig. 1. Schemes of the beam investigated](image)

A) Cantilever bilayered composite beam with different moduli in the two crack arms

B) Cantilever bilayered composite beam with different moments of inertia in the two crack arms

The purpose of the present work is the derivation of formula for the strain energy release rate, $G$, in the case of crack in cantilever bilayered composite beam. In order to perform that, the result obtained on the basis of the simple beam theory according to the linear elastic fracture mechanics hypotheses will be used [12]. The solution of the problem includes determination of the beam characteristics in the un-cracked portion as functions of the two crack arms properties.
2. Determination of the strain energy release rate, $G$

Using the conventional linear elastic beam theory the authors have derived the following equation for $G$ [12]:

\[
G = \frac{1}{2Eb} \left( \frac{N_1^2}{A_1} + \frac{N_2^2}{A_2} - \frac{N^2}{A} \right) + \frac{1}{2Eb} \left( \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} - \frac{M^2}{I} \right) + \frac{1}{2G'b} \left( k \frac{V_1^2}{A_1} + k \frac{V_2^2}{A_2} - k \frac{V^2}{A} \right),
\]

where $N_1, V_1, M_1$ are the internal forces in the upper arm of the crack behind its tip, $N_2, V_2, M_2$ are the internal forces in the lower arm of the crack behind its tip, $N, V, M$ are the internal forces in front of the crack tip; $A_1, A_2, A$ are the areas, while $I_1, I_2, I$ are the moments of inertia of the cross-sections in front of and behind the crack tip; $E$ and $G'$ are the modulus of elasticity and the shear modulus, respectively, of the material building the beam; $k$ is the shear coefficient.

According to Eq. (1), the internal forces caused by the force $F$ have to be obtained first. After that, the properties of the beam in the un-cracked portion must be determined. Finally, the results derived have to be substituted in Eq. (1).

### 2.1 Cantilever bilayered composite beam with different moduli in the two crack arms

The beam is built by two different unidirectional fiber reinforced composite materials. The crack is situated between them.

To find the internal forces in the two arms of the crack behind its tip the external force is resolved into components:

\[
F = F_1 + F_2.
\]

Applying the equilibrium equations for each of the crack arms the internal forces are derived (Fig. 2):

\[
N_1 = 0, \quad V_1 = -F_1, \quad M_1 = -F_1a; \tag{3}
\]

\[
N_2 = 0, \quad V_2 = -F_2, \quad M_2 = -F_2a. \tag{4}
\]

The internal forces in front of the crack tip are (Fig. 3):

\[
N = 0, \quad V = -F, \quad M = -Fa. \tag{5}
\]
Then, using the condition that the crack breaks the connection between the layers, the bending moments in the two crack arms are expressed as functions of the bending moment in front of the crack tip [13]:

\[ M_1 = M - \frac{E_1 I_1}{E_1 I_1 + E_2 I_2} = -F a E_1, \]

\[ M_2 = M - \frac{E_2 I_2}{E_1 I_1 + E_2 I_2} = -F a E_2. \]

Here, \( I_1 = I_2 = \frac{bh^3}{12} \).

Further, taking the expressions of the bending moments from Eqs. (3), (4) and (5) and replacing them in Eqs. (6) and (7), the components \( F_1 \) and \( F_2 \) are determined as:

\[ F_1 = F \frac{E_1}{E_1 + E_2}, \]

\[ F_2 = F \frac{E_2}{E_1 + E_2}. \]
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It should be noted that Eqs. (8) and (9) might be also derived using the assumption that the crack arms mutual displacements in the transverse direction are equal to zero.

Finally, all of the internal forces are obtained as functions of the force $F$:

$$ V_1 = -F \frac{E_1}{E_1 + E_2}, \quad V_2 = -F \frac{E_2}{E_1 + E_2}, \quad V = -F; $$

$$ M_1 = -Fa \frac{E_1}{E_1 + E_2}, \quad M_2 = -Fa \frac{E_2}{E_1 + E_2}, \quad M = -Fa. $$

Next step in the solution is determination of the modulus of elasticity and shear modulus for the un-cracked beam part. In accordance with [14], the stiffness of the bilayered beam subjected to bending is:

$$ EI = E_1 I_1 + E_2 I_2 + \frac{E_1 A_1 E_2 A_2 c^2}{E_1 A_1 + E_2 A_2}, $$

where $E$ is the modulus of elasticity of the un-cracked beam, $E_1$ and $E_2$ are the moduli of elasticity of the two layers, respectively, $c = h$ is the distance between the two layers’ centers of gravity. Besides, $I = \frac{2bh^3}{3}$, $A_1 = A_2 = bh$. Thus, from Eq. (12) one may obtain:

$$ E = \frac{E_1^2 + E_2^2 + 14E_1 E_2}{8(E_1 + E_2)}. $$

The shear modulus of the un-cracked beam is given by the expression [15]:

$$ G' = \frac{2G'_1 G'_2}{G'_1 + G'_2}, $$

where $G'_1$ and $G'_2$ are the shear moduli of the two layers, respectively.

Furthermore, to derive the formula for the strain energy release rate Eq. (1) is rearranged in the manner:

$$ G = \frac{1}{2b} \left( \frac{M_1^2}{E_1 I_1} + \frac{M_2^2}{E_2 I_2} - \frac{M'^2}{E I} \right) + \frac{1}{2b} \left( k \frac{V_1^2}{G'_1 A_1} + k \frac{V_2^2}{G'_2 A_2} - k \frac{V'^2}{G'A} \right) = $$

$$ = \frac{1}{2b} \left( \frac{12M_1^2}{E_1 bh^3} + \frac{12M_2^2}{E_2 bh^3} - \frac{3M'^2}{2E bh^3} \right) + \frac{1}{2b} \left( k \frac{V_1^2}{G'_1 bh} + k \frac{V_2^2}{G'_2 bh} - k \frac{V'^2}{2G'A} \right) = $$

$$ = \frac{6}{b^2 h^3} \left( \frac{M_1^2}{E_1} + \frac{M_2^2}{E_2} - \frac{M'^2}{E} \right) + k \frac{V_1^2}{G'_1} + \frac{V_2^2}{G'_2} \left( \frac{V'^2}{2G'} \right). $$
Then, Eqs. (10), (11), (13) and (14) are substituted in Eq. (15):

\[
G = \frac{6}{b^2 h^3} \left[ \frac{1}{E_1} \left( -Fa \frac{E_1}{E_1 + E_2} \right)^2 + \frac{1}{E_2} \left( -Fa \frac{E_2}{E_1 + E_2} \right)^2 - \frac{8(E_1 + E_2)}{8(E_1^2 + E_2^2 + 14E_1E_2)(-F a)^2} \right] + \\
+ \frac{k}{2b^2 h} \left[ \frac{1}{G'_1} \left( -F \frac{E_1}{E_1 + E_2} \right)^2 + \frac{1}{G'_2} \left( -F \frac{E_2}{E_1 + E_2} \right)^2 - \frac{G'_1 + G'_2}{4G'_1G'_2} \right].
\]

Finally, the following equation for \( G \) is obtained:

\[
G = \frac{72F^2a^2}{b^2 h^3} \left[ \frac{E_1 E_2}{(E_1 + E_2)(E_1^2 + E_2^2 + 14E_1E_2)} \right] + \\
+ \frac{kF^2}{2b^2 h} \left[ \frac{1}{(E_1 + E_2)^2} \left( \frac{E_1^2}{G'_1} + \frac{E_2^2}{G'_2} \right) - \frac{G'_1 + G'_2}{4G'_1G'_2} \right].
\]

### 2.2. Cantilever bilayered composite beam with different moments of inertia in the two crack arms

Here, the beam is made of one unidirectional fiber reinforced composite material, but two layers have different heights, moments of inertia and areas of the cross-sections. The geometrical characteristics are (Fig. 1B):

\[
I_1 = \frac{bh_1^3}{12}, \quad I_2 = \frac{bh_2^3}{12}, \quad I = \frac{2}{3}bh^3 = \frac{b(h_1 + h_2)^3}{12};
\]

\[
A_1 = bh_1, \quad A_2 = bh_2, \quad A = 2bh = b(h_1 + h_2);
\]

\[
h = \frac{1}{2}(h_1 + h_2).
\]

To determine the internal forces in the two arms of the crack behind its tip the external force is resolved into components again:

\[
F = F_1 + F_2.
\]

Using the equilibrium equations for the crack arms the internal forces are obtained (Fig. 4):

\[
N_1 = 0, \quad V_1 = -F_1, \quad M_1 = -F_1a;
\]
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Fig. 4. Internal forces behind the crack tip in the case of bilayered composite beam with different moments of inertia in the two crack arms

Fig. 5. Internal forces in front of the crack tip in the case of bilayered composite beam with different moments of inertia in the two crack arms

\[(21)\]
\[
N_2 = 0, \quad V_2 = -F_2, \quad M_2 = -F_2a.
\]

The internal forces in front of the crack tip are (Fig. 5):

\[(22)\]
\[
N = 0, \quad V = -F, \quad M = -Fa.
\]

The bending moments in the crack arms are expressed as functions of the bending moment in front of the crack tip [13]:

\[(23)\]
\[
M_1 = M \frac{E_1 I_1}{E_1 I_1 + E_2 I_2} = -Fa \frac{I_1}{I_1 + I_2} = -Fa \frac{h_1^3}{h_1^3 + h_2^3},
\]

\[(24)\]
\[
M_2 = M \frac{E_2 I_2}{E_1 I_1 + E_2 I_2} = -Fa \frac{I_2}{I_1 + I_2} = -Fa \frac{h_2^3}{h_1^3 + h_2^3},
\]

where \(E_1 = E_2 = E\).

Then, the components \(F_1\) and \(F_2\) of the external force are:

\[(25)\]
\[
F_1 = F \frac{h_1^3}{h_1^3 + h_2^3},
\]
Thus, the internal forces are obtained as a function of force $F$:

$$V_1 = -F \frac{h_3^1}{h_1^3 + h_2^3}, \quad V_2 = -F \frac{h_3^2}{h_1^3 + h_2^3}, \quad V = -F;$$

$$M_1 = -Fa \frac{h_3^1}{h_1^3 + h_2^3}, \quad M_2 = -Fa \frac{h_3^2}{h_1^3 + h_2^3}, \quad M = -Fa.$$
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which will be used for the check of Eqs. (17) and (30).

3.1 Cantilever bilayered composite beam with different moduli in the two crack arms

To perform the check the moduli $E = E_1 = E_2$ and $G' = G'_1 = G'_2$ are substituted in Eq. (17). The result is:

$$\begin{align*}
G &= \frac{6F^2a^2}{b^2h^3} \left[ \frac{12EE}{(E+E)(E^2+E^2+14EE)} \right] + \\
&\quad + \frac{kF^2}{2b^2h} \left[ \frac{1}{(E+E)^2} \left( \frac{E^2}{G'} + \frac{E^2}{G'} \right) - \frac{G' + G'}{4G'G'} \right] = \frac{9F^2a^2}{4Eb^2h^5}.
\end{align*}$$

3.2 Cantilever bilayered composite beam with different moments of inertia in the two crack arms

Now, $h = h_1 = h_2$ is substituted in Eq. (30). The result is:

$$\begin{align*}
G &= \frac{6F^2a^2}{Eb^2} \left[ \frac{1}{h^3 + h^3} - \frac{1}{(h+h)^3} \right] + \\
&\quad + \frac{kF^2}{2G'b^2} \left( \frac{h^5 + h^5}{(h^3 + h^3)^2} - \frac{1}{h+h} \right) = \frac{9F^2a^2}{4Eb^2h^5}.
\end{align*}$$

It is obvious that the equations (32) and (33) match completely to formula (31) which shows that the expressions derived for the strain energy release rate are correct and may be successfully used to investigate the cracks in bilayered composite beams.

4. Investigation on the influence of the moduli of elasticity ratio and the moments of inertia ratio on the strain energy release rate

4.1 Cantilever bilayered composite beam with different moduli in the two crack arms

To investigate the influence of the moduli of elasticity ratio on the strain energy release rate formula (17) is rearranged in the following manner:

$$\begin{align*}
\frac{G}{E_1b} &= \frac{72F^2a^2}{E_1b^2h^3} \left[ \frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)} \right] + \\
&\quad + \frac{kF^2}{2E_1b^2h} \left[ \frac{1}{(E_1+E_2)^2} \left( \frac{E_2^2}{G'_1} + \frac{E_2^2}{G'_2} \right) - \frac{G'_1 + G'_2}{4G'_1G'_2} \right].
\end{align*}$$
Then, the coefficients \( \theta = \frac{E_1}{E_2} \) and \( \eta = \frac{G_1'}{G_2} \) are introduced:

\[
(35) \quad \frac{G}{E_1 b} = \frac{72F^2\alpha^2}{E_1^2 b^5 h^3} \left[ \frac{\theta^2}{(1 + \theta)(1 + \theta^2 + 14\theta)} \right] + \\
+ \frac{kF^2}{2E_1 G_1' b^3 h} \left[ \frac{\theta^2 + \eta}{(1 + \theta)^2} - \frac{1 + \eta}{4} \right].
\]

After that, the parameter \( \theta \) is assumed to vary in interval \([1:9]\) while \( \eta \) and the expressions out of the brackets are taken to be equal to one.

The diagram for the influence of the coefficient \( \theta \) on the strain energy release rate \( G \) is given in Fig. 6. It may be observed that, if \( \theta \) increases, \( G \) increases, too. This finding could be explained by the fact that when \( \theta \) increases the lower crack arm stiffness decreases. Thus, the stiffness of entire beam decreases which leads to increasing of \( G \).

Fig. 6. Diagram for the influence of the moduli of elasticity ratio on the strain energy release rate

**4.2 Cantilever bilayered composite beam with different moments of inertia in the two crack arms**

Expression (30) is written in the form:

\[
(36) \quad \frac{G}{Eb} = \frac{6F^2\alpha^2}{E^2 b^3} \left[ \frac{1}{h_1^3 + h_2^3} - \frac{1}{(h_1 + h_2)^3} \right] + \\
+ \frac{kF^2}{2EG'b^5} \left[ \frac{h_1^5 + h_2^5}{(h_1^3 + h_2^3)^2} - \frac{1}{h_1 + h_2} \right].
\]
After that, the parameter \( \xi = \frac{I_1}{I_2} = \frac{h_3^1}{h_3^2} \) is introduced:

\[
(37) \quad \frac{G}{Eb} = \frac{6F^2a^2}{E^2b^3h_1^3} \left[ \frac{\xi}{1 + \xi} - \frac{1}{(1 + \sqrt[3]{\xi})^3} \right] + \\
+ \frac{kF^2}{2EG'b^3h_1} \left\{ \xi^2 \left[ \frac{1 + (\sqrt[3]{\xi})^5}{(1 + \xi)^2 (\sqrt[3]{\xi})^5} - \frac{\sqrt[3]{\xi}}{1 + \sqrt[3]{\xi}} \right] \right\}.
\]

Further, the variation of \( \xi \) in interval \([1 \div 5]\) is defined and the expressions out of the brackets are assumed to be equal to one.

The diagram for the influence of \( \xi \) on the strain energy release rate is shown in Fig. 7. It is obvious, that when the crack arms have equal moments of inertia, \( G \) possesses minimum value. This is due to the fact that when the moments of inertia are equal, the difference between the crack tip longitudinal displacements in the two arms is smallest.

![Fig. 7. Diagram for the influence of the moments of inertia ratio on the strain energy release rate](image)

**5. Conclusions**

In the present study the mode II crack problem in cantilever bilayered composite beam is considered. The closed form analytical solution for the strain energy release rate is obtained using the conventional linear elastic beam theory. The derived formula for \( G \) is verified by performing comparisons with a known analytical solution for a homogeneous cantilever beam with equal heights of the crack arms. The influence of the moduli of elasticity ratio as well as the two crack arms moments of inertia ratio on the strain energy release
rate is investigated. It is shown that when these ratios increase the strain energy release rate increases, too. This finding is explained with increase of the difference between the crack tip longitudinal displacements in the two arms.

REFERENCES