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APPLICATION OF J-INTEGRAL IN THE CASE OF A SINGLE CRACK IN CANTILEVER BEAM*

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Abstract. In the present study the J-integral approach has been applied to investigate a single crack in two cantilever beams made of unidirectional fiber reinforced composite. The crack is situated in the middle of the beam's cross-section and is parallel to the reinforcing fibers. The two beams are loaded in different manner, but both loading configurations are asymmetric with respect to the crack. Closed form analytical solutions of the J-integral have been obtained using the linear-elastic beam theory. It was established that in the cases under consideration the term containing the strain energy density is not equal to zero and exerts a great influence on the J-integral value. Comparisons between the J-integral expressions and the formulas for the strain energy release rate are performed and a very good agreement is obtained. The dependence of the J-integral magnitude on the crack length has been also examined and the results obtained are presented graphically. The present paper is a part of an investigation in the field of fracture behaviour of fiber reinforced composite beams.
Key words: Crack, $J-$integral, fracture mechanics, cantilever beam.

1. Introduction

Fracture mechanics studies the failure in the construction elements due to crack initiation and propagation [1, 2]. One of the basic approaches in fracture mechanics is J-integral, which is a powerful tool for analysis of cracks in non-linear materials [3, 4, 5].

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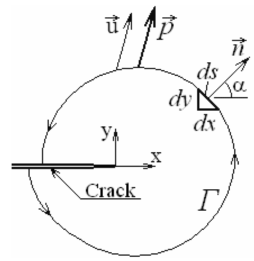

Fig. 1. The J-integral components

J-integral is defined by expression:

$$J = \int_{\Gamma} \left\{ u_0 \cos\alpha - \left[p_x \frac{\partial u}{\partial x} + p_y \frac{\partial v}{\partial x} \right] \right\} ds, \tag{1}$$

where u_0 is the strain energy density, p_x and p_y are the components of the traction vector on the contour Γ, u and v are the displacements along the axes x and y, respectively, ds is the infinitesimal part of the contour, α is the angle between the normal toward the contour and the x-axis (Fig. 1). Integration starts from the lower crack arm, goes along the contour Γ in the counter clockwise direction and finishes in the upper crack arm. It is very important that the J-integral value does not depend on the chosen integration path [6, 7, 8]. It should be noted that the J-integral can also be used in the case of linear-elastic materials where its value is equal to the value of the strain energy release rate, G.

In this study, the objects of investigation are two cantilever beams made of unidirectional fiber reinforced composite [9, 10, 11]. The beams subjected to different loading configurations have rectangular cross-sections and contain a crack of length a (Fig. 2). It should be mentioned that in the literature available the authors have not found solutions of J-integral for the cases considered.

2. *J*-integral solution

2.1. Case 1

The load is a vertical force F applied to the cross-section A of the lower crack arm. The support reactions in cross-section B as well as the integration contour are shown in Fig. 3.

The expression of the J-integral is:

$$J = J_A + J_B. \tag{2}$$

Here, J_A is the J-integral value in cross-section A, while J_B is the value in the fixed support. In the other contour segments the J-integral value is equal to zero.

• **Determination of J_A**

The expression of J_A has the following form:

$$J_A = \int\limits_{\Gamma} \left\{ - \left[p_{y_A} \frac{\partial v_A}{\partial x} \right] \right\} ds_A, \tag{3}$$

where $ds_A = dx$ and p_{y_A} is the normal stress on the segment $-a < x < -a + l$ in cross-section A caused by the force F:

$$p_{y_A} = -\frac{F}{bl}. \tag{4}$$

In the present investigation the partial derivative $\frac{\partial v_A}{\partial x}$ is replaced by $\frac{dv_A}{dx}$ because v_A is a function of x only. The value of $\frac{dv_A}{dx}$ is equal to the slope of cross-section A (Fig. 4):

$$\frac{dv_A}{dx} = \frac{6F}{EbH^3} \left(7a^2 + L^2\right). \tag{5}$$

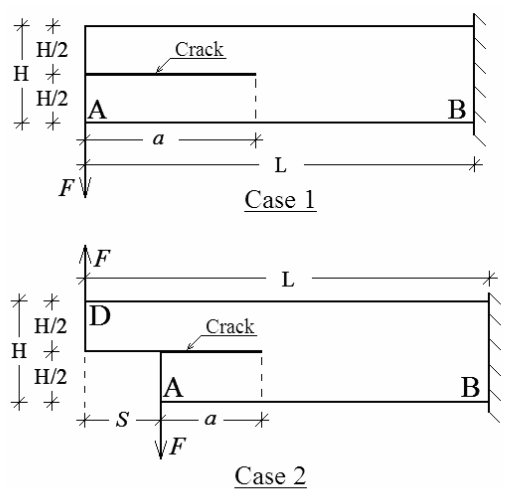

Fig. 2. Cantilever beams

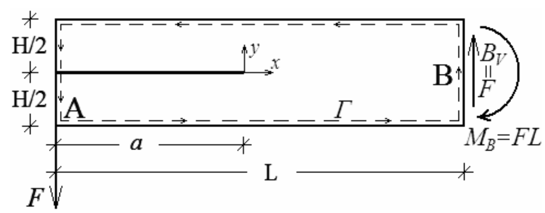

Fig. 3. Loading and integration contour Γ – case 1

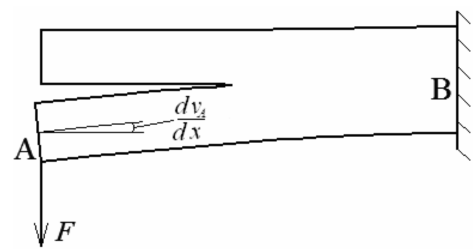

Fig. 4. Deformation of the beam investigated – case 1

After substitution of Eqs. (4) and (5) in (3) we obtain:

$$J_A = \int_{-a}^{-a+l} \left\{ - \left(-\frac{F}{bl} \right) \frac{6F}{EbH^3} \left(7a^2 + L^2 \right) \right\} dx = \frac{6F^2}{Eb^2H^3} \left(7a^2 + L^2 \right). \tag{6}$$

• **Determination of J_B**

The expression of J_B is:

$$J_B = \int_{\Gamma} \left\{ u_{0_B} \cos\alpha - \left[p_{x_B} \frac{\partial u_B}{\partial x} \right] \right\} ds_B, \tag{7}$$

where u_{0_B} is the strain energy density in cross-section B while p_{x_B} and $\frac{\partial u_B}{\partial x}$ are the normal stress and the linear strain in cross-section B, respectively. Besides, $ds_B = dy$ and $\cos\alpha = 1$, because angle α is equal to $0°$. Furthermore, only the bending moment in the cross-section B is taken into account in the calculations, because long beams ($L \gg H$) are considered in the present paper. Thus, the strain energy density is:

$$u_{0_B} = \frac{\sigma_{x_B}^2}{2E}. \tag{8}$$

Here, σ_{x_B} is the normal stress in cross-section B and E is the modulus of elasticity.

The normal stress is obtained by the well-known formula of Navier [12]:

$$\sigma_{x_B} = \frac{M_B}{I} y = \frac{-FL}{I} y. \tag{9}$$

Thus, from Eqs. (8) and (9) we obtain:

$$u_{0_B} = \frac{1}{2E} \left[\frac{-FL}{I} y \right]^2 = \frac{F^2 L^2}{2EI^2} y^2. \tag{10}$$

After that, p_{x_B} is determined by Eq. (9), since $p_{x_B} \equiv \sigma_{x_B}$. The partial derivative $\dfrac{\partial u_B}{\partial x}$ is the first order derivative of the longitudinal displacement function, $u(x) = \dfrac{M}{EI} xy = \dfrac{-FL}{EI} xy$ [13, 14]:

$$\frac{\partial u_B}{\partial x} = \frac{-FL}{EI} y. \tag{11}$$

Further, we substitute Eqs. (9), (10) and (11) in Eq. (7):

$$J_B = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ \left[\frac{F^2 L^2}{2EI^2} y^2 \right] - \left[-\frac{FL}{I} y \left(-\frac{FL}{EI} y \right) \right] \right\} dy = \tag{12}$$

$$= \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ \left[-\frac{F^2 L^2}{2EI^2} y^2 \right] \right\} dy = -\frac{6F^2 L^2}{Eb^2 H^3}.$$

Finally, the solution of J-integral is obtained by substitution of Eqs. (6) and (12) in Eq. (2):

$$J = \left[\frac{6F^2}{Eb^2 H^3} \left(7a^2 + L^2 \right) \right] + \left[-\frac{6F^2 L^2}{Eb^2 H^3} \right] = \frac{42 F^2 a^2}{Eb^2 H^3}. \tag{13}$$

2.2. Case 2

The loading consists of two opposite vertical forces applied to the cross-sections A and D of the two crack arms. Besides, the crack begins at a distance S from the cross-section D (Fig. 5).

The J-integral value is expressed as:

$$J = J_A + J_B + J_D, \tag{14}$$

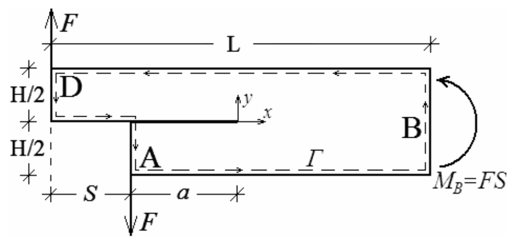

Fig. 5. Loading and integration contour Γ – case 2

where J_A is the value of J-integral in cross-section A, J_B is the value in the fixed support, and J_D is the value in cross-section D. The $J-$integral value is equal to zero in the other segments of the integration contour.

• **Determination of J_A**

Here, the determination of J_A is similar to this one in the previous case. However, there is a difference in the determination of $\dfrac{dv_A}{dx}$ (Fig. 6).

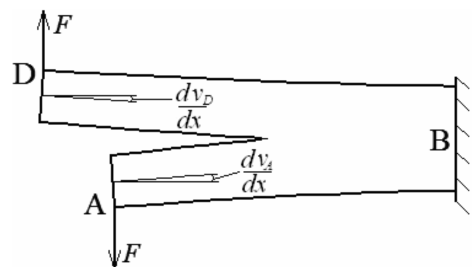

Fig. 6. Deformation of the beam investigated – case 2

$$J_A = \int_\Gamma \left\{ - \left[p_{y_A} \frac{dv_A}{dx} \right] \right\} ds_A; \tag{15}$$

$$p_{y_A} = -\frac{F}{bl}; \tag{16}$$

$$\frac{dv_A}{dx} = \frac{12F}{EbH^3} \left[4a^2 - S(L - S - a) \right]; \tag{17}$$

$$J_A = \int_{-a}^{-a+l} \left\{ -\left(-\frac{F}{bl} \right) \frac{12F}{EbH^3} \left[4a^2 - S(L-S-a) \right] \right\} dx = \\ = \frac{12F^2}{Eb^2H^3} \left[4a^2 - S(L-S-a) \right]. \quad (18)$$

• **Determination of J_B**

The expression of J_B is identical with J_B in the case 1.

$$J_B = \int_\Gamma \left\{ u_{0_B} \cos\alpha - \left[p_{x_B} \frac{\partial u_B}{\partial x} \right] \right\} ds_B. \quad (19)$$

Here, $ds_B = dy$ and $\cos\alpha = 1$.

The bending moment acting in beam cross-section B causes the normal stresses. The strain energy density is presented below:

$$u_{0_B} = \frac{\sigma_{x_B}^2}{2E}, \quad (20)$$

where

$$p_{x_B} \equiv \sigma_{x_B} = \frac{M_B}{I} y = \frac{FS}{I} y. \quad (21)$$

After that:

$$u_{0_B} = \frac{1}{2E} \left[\frac{FS}{I} y \right]^2 = \frac{F^2S^2}{2EI^2} y^2. \quad (22)$$

Further

$$\frac{\partial u_B}{\partial x} = \frac{FS}{EI} y. \quad (23)$$

Finally:

$$J_B = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ \left[\frac{F^2S^2}{2EI^2} y^2 \right] - \left[\frac{FS}{I} y \left(\frac{FS}{EI} y \right) \right] \right\} dy = \\ = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ \left[-\frac{F^2S^2}{2EI^2} y^2 \right] \right\} dy = -\frac{6F^2S^2}{Eb^2H^3}. \quad (24)$$

• **Determination of J_D**

Expression of J_D is:

$$J_D = \int\limits_{\Gamma} \left\{ - \left[p_{y_D} \frac{dv_D}{dx} \right] \right\} ds_D, \tag{25}$$

where $ds_D = -dx$ and p_{y_D} is the normal stress in the segment $-S-a+l < x < -S-a$ in cross-section D caused by the force F:

$$p_{y_A} = \frac{F}{bl}. \tag{26}$$

The derivative $\frac{dv_D}{dx}$ is equal to slope of section D (Fig. 6):

$$\frac{dv_D}{dx} = -\frac{12F}{EbH^3} \left[4(S+a)^2 + S(L-S-a) \right]. \tag{27}$$

We obtain for J_D:

$$J_D = \int_{-S-a+l}^{-S-a} \left\{ - \left(\frac{F}{bl} \right) \left(-\frac{12F}{EbH^3} [4(S+a)^2 + S(L-S-a)] \right) \right\} (-dx) = \\ = \frac{12F^2}{Eb^2H^3} \left[4(S+a)^2 + S(L-S-a) \right]. \tag{28}$$

Finally, substituting Eqs. (19), (25) and (29) in Eq. (15) we obtain:

$$J = \left\{ \frac{12F^2}{Eb^2H^3} \left[4a^2 - S(L-S-a) \right] \right\} + \left\{ -\frac{6F^2S^2}{Eb^2H^3} \right\} + \\ + \left\{ \frac{12F^2}{Eb^2H^3} \left[4(S+a)^2 + S(L-S-a) \right] \right\} = \frac{6F^2}{Eb^2H^3} \left(16a^2 + 16Sa + 7S^2 \right). \tag{29}$$

3. Verification of the solution

The formula for the strain energy release rate G obtained in [15] will be used to check expressions (13) and (29). It has been already said, that in the case of the linear elastic material the value of G is equal to the J-integral value. The formula for G is:

$$G = \frac{1}{2Eb}\left(\frac{N_1^2}{A_1} + \frac{N_2^2}{A_2} - \frac{N^2}{A}\right) + \frac{1}{2Eb}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} - \frac{M^2}{I}\right) + + \frac{1}{2G_1 b}\left(k\frac{V_1^2}{A_1} + k\frac{V_2^2}{A_2} - k\frac{V^2}{A}\right), \tag{30}$$

where N_1, V_1, M_1 are the internal forces behind the crack tip in the upper arm, N_2, V_2, M_2 are the internal forces in the lower crack arm, N, V, M are the internal forces in front of the crack tip, k is the shear coefficient. A_1, A_2, A are the cross-sectional areas behind and in front of the crack tip, I_1, I_2, I are the moments of inertia, E and G_1 are the modulus of elasticity and shear modulus, respectively.

3.1. Case 1

For this loading configuration we have:

$$\begin{array}{lll} N_1 = 0, & N_2 = 0, & N = 0, \\ V_1 = 0, & V_2 = -F, & V = -F, \\ M_1 = 0, & M_2 = -Fa, & M = -Fa. \end{array} \tag{31}$$

By substituting Eqs. (31) in Eq. (30) we obtain:

$$G = \frac{42F^2a^2}{Eb^2H^3}. \tag{32}$$

It should be specified that the shearing forces are neglected in Eq. (32), because as already mentioned long beams are considered in the present paper.

3.2. Case 2

Here:

$$\begin{array}{lll} N_1 = 0, & N_2 = 0, & N = 0, \\ V_1 = F, & V_2 = -F, & V = 0, \\ M_1 = F(S+a), & M_2 = Fa, & M = FS. \end{array} \tag{33}$$

Then, we obtain:

$$G = \frac{6F^2}{Eb^2H^3}\left(16a^2 + 16Sa + 7S^2\right). \tag{34}$$

It is obvious, that Eqs (32) and (34) completely match the results obtained for the J-integral in both cases. This is an indication for the correctness of the solutions obtained.

4. Influence of the crack length on the J-integral value

4.1. Case 1

Expression (13) is rearranged in order to perform this investigation, in a way presented below:

$$\frac{J}{Eb} = \frac{42F^2a^2}{E^2b^3H^3}. \tag{35}$$

The influence of the crack length a on the $J-$integral is illustrated in Fig. 7. It is obvious that the increasing of a leads to increasing of J. This finding can be attributed to the increasing of the bending moment in the lower crack arm.

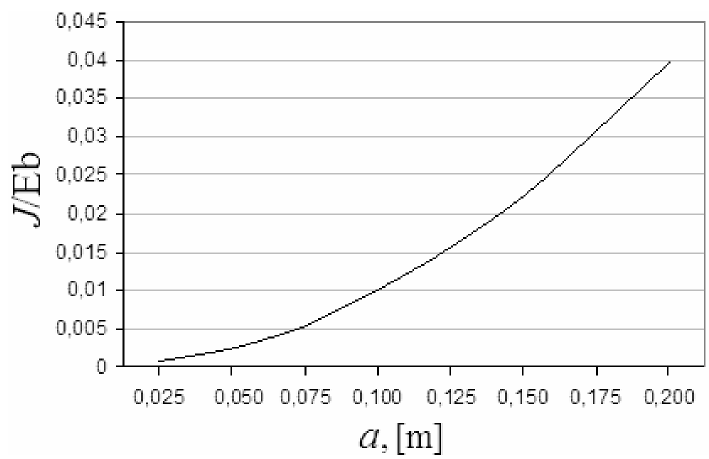

Fig. 7. Diagram of the relation between J-integral value and the crack length a – case 1

4.2. Case 2

Equation (29) is written in the form:

$$\frac{J}{Eb} = \frac{6F^2}{Eb^2H^3}\left(16a^2 + 16Sa + 7S^2\right). \tag{36}$$

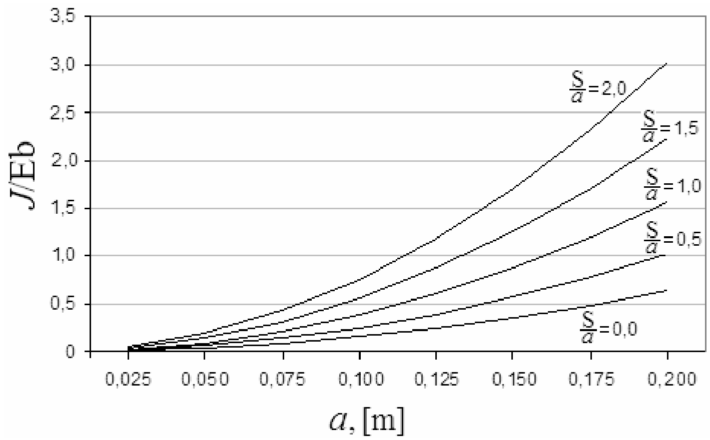

Fig. 8. Diagram of the relation between J-integral value and the crack length a for several ratios $\dfrac{S}{a}$ – case 2

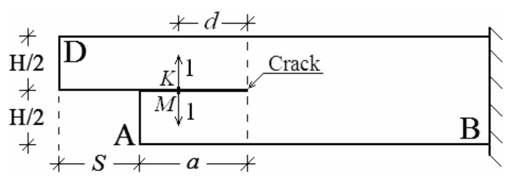

Fig. 9. Virtual loading used for determination of the two crack arms mutual displacements in vertical direction

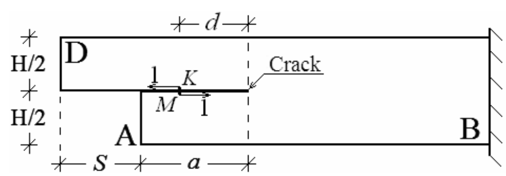

Fig. 10. Virtual loading used for determination of the two crack arms mutual displacements in horizontal direction

Expression (36) shows that the J-integral magnitude depends on the crack length a and on the distance S, as well. The relation between J-integral and crack length a in the cases of different values of the ratio $\dfrac{S}{a}$ is given in Fig. 8. It can be observed that the bigger $\dfrac{S}{a}$, the bigger J. This circumstance could be explained by determination of the mutual displacements in vertical and horizontal directions of two points (K and M) belonging to the crack arms caused by the forces F. The corresponding virtual loading is shown in Figs 9 and 10. The result is:

$$\Delta_V = \frac{16F}{EbH^3}\left\{3d\left[(S+a)^2+a^2\right]-\left[(S+a)^3+a^3\right]\right\}, \tag{37}$$

$$\Delta_H = \frac{12F}{EbH^2}(S^2+2Sa), \tag{38}$$

where Δ_V and Δ_H are the mutual displacements in the vertical and horizontal directions, respectively, d is the distance from points K and M to the crack tip $(0 < d \leq a)$.

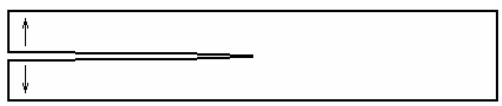

Fig. 11. Mode I fracture

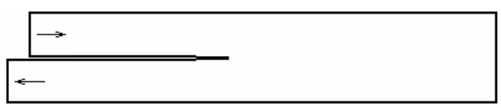

Fig. 12. Mode II fracture

The analysis about the influence of the crack arms mutual displacements on J-integral begins with formula (37), which shows that $\Delta_V > 0$ for every value of distance S. This means that in case 2 of the cantilever beam the Mode I fracture always exists (Fig. 11). It can be observed from Eq. (38) that if $S = 0$, Δ_H is also zero. This indicates that there is no Mode II fracture, since the mutual crack displacements parallel to the crack direction corresponds to Mode II (Fig. 12). The mutual displacement of the crack arms in longitudinal direction is different than zero when $S > 0$, i.e. Mode II fracture arises leading to the Mixed mode I/II crack growth. Actually, Mode II fracture causes increasing of J-integral with increasing of distance S.

5. Conclusions

It has been proved in the study that J-integral is a powerful method for crack analysis. The application of J-integral is facilitated by the fact that its value is independent on the integration contour. Therefore, in the present analysis the integration contour was chosen to coincide with the beam contour which substantially made easier obtaining the solutions.

It has been shown that for the configurations considered fracture behaviour can be successfully analyzed using the conventional linear-elastic beam theory. The validity of the J-integral solutions obtained has been proved by comparisons with expressions for the strain energy release rate, G. From practical point of view, the analytical solutions of J-integral obtained in the present work are very useful for parametrical study of fracture behaviour. Such studies can be used for optimization of composite beams with respect to their fracture resistance.

REFERENCES

[1] Siratori, M., T. Miesi, H. Matsusita. Computational Fracture Mechanics, Moscow, Mir, 1986 (in Russian).
[2] Ewalds, H. L., R. J. H. Wanhill. Fracture Mechanics, Hodder Headline PLC, 1984.
[3] Morozov, N. Mathematical Aspects of Fracture Mechanics, Moscow, Nauka, 1984 (in Russian).
[4] Matvienko, Y. Fracture Mechanics Models and Criteria, Moscow, Fizmatlit, 2006 (in Russian).
[5] Broek, D. Elementary Engineering Fracture Mechanics, Martinus Nijhoff Publishers, 1982.
[6] Erdogan, F., A. Kobayasi, S. Atlury et al. Computational Methods in the Mechanics of Fracture, Moscow, Mir, 1990 (in Russian).
[7] Cherepanov, G. Fracture Mechanics in Composite Materials, Moscow, Nauka, 1983 (in Russian).
[8] Liebowitz, H. Fracture – volume 2 – Mathematical Fundamentals, Moscow, Mir, 1975 (in Russian).
[9] Pilkey, W. D. Formulas for Stress, Strain and Structural Matrices, John Wiley&Sons, 1994.
[10] Choo, V. Fundamentals of Composite Materials, Knowen Academic Press, 1990.
[11] Pobedrya, B. Mechanics of Composite Materials, Moscow University, 1984 (in Russian).

[12] Kislyakov, S. Strength of Materials, Sofia, Tehnika, 1975 (in Bulgarian).

[13] Aleksandrov, A., V. Potapov. Strength of Materials: Fundamentals of the Theory of Elasticity and Plasticity, Moscow, Vizshaya shkola, 2002 (in Russian).

[14] Varbanov, C. Theory of Elasticity, Sofia, Tehnika, 1962 (in Bulgarian).

[15] Mladensky, A., V. Rizov. Determination of the Strain Energy Release Rate at Crack Propagation in Composite Materials, Annual of UACG, Volume XLII (2002–2003), Facsimile II, 129–143 (in Bulgarian).