

FLUID MECHANICS

STATIC EQUILIBRIUM OF THIN FILMS ON AXISYMMETRIC SUBSTRATES

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ABSTRACT. The static equilibrium of a thin liquid film spread on a rigid axisymmetrical substrate is studied. A curvilinear coordinate system fitted to the substrate is introduced. Three axisymmetrical types of substrate shapes are considered: plane, spherical and arbitrary. The shapes on the films on the spherical and arbitrary axisymmetrical substrates are determined numerically for different Bond numbers, wetting angles and film volumes. Shape bifurcations are observed leading to film instability for some combinations of these parameters. In the case of a plane substrate, the film is always at stable equilibrium.

KEY WORDS: thin film, static equilibrium, axisymmetrical substrate, stability, bifurcation.

1. Introduction

The study of the thin liquid film behaviour is related to a great number of natural and technological processes, where thin films can be found, e.g. coalescence of drops, thin film layer coatings, icicle formation, etc. Usually, the problems become extremely difficult, when the capillary forces action is important and if there is a moving contact line between the fluid and the solid surface. However, when the processes are very slow, it is possible to be treated as quasi-static or static. The capillary statics problems have been successfully investigated for more than two centuries [1], [2]. It occurs, also, that the spreading of axisymmetrical drops on flat surfaces, which leads to thin film formation is not continuous, but reaches the drop equilibrium sizes and shapes [3]. The dimensionless radius of the deformed drop and the drop shape are

functions of the Bond number (ratio between the gravity and capillary forces) and are expressed by analytical formulas, if the wetting angle is small. The problem becomes non-linear for bigger wetting angles and only numerically found shapes exist, which will be shown in the present work. We shall investigate thin films also on arbitrary curved axisymmetrical substrates, as they are found in draining, coating and biological flows. The effects of the curvature of the substrate are in competition with the gravity and capillary forces and their analysis occurs important. The curvature effects on dynamic models, such as lubrication model, have been presented in [4] and [5]. In the latter paper, three possible cases for the substrate curvature are considered: the substrate is almost plane and its curvature enters the lubrication equation as a body force; the substrate curvature is constant with a large magnitude, which corresponds to a destabilizing term in the lubrication equation; the curvature radius is large but not constant.

In the present work the film liquid is supposed incompressible and newtonian and the film reaches its static state at a prescribed wetting angle with the solid substrate. Three types of axisymmetric substrates (as formed after drop impact onto another drop) are considered as shown in Fig. 1: plane substrate with zero curvature; spherical or truncated spherical substrate with constant curvature and arbitrary (shape taken from experiments) substrate with variable curvature. The film shape is found analytically or numerically from the Laplace-Young equation written in general coordinates fitted to the substrate, so called body fitted coordinates, for any Bond number.

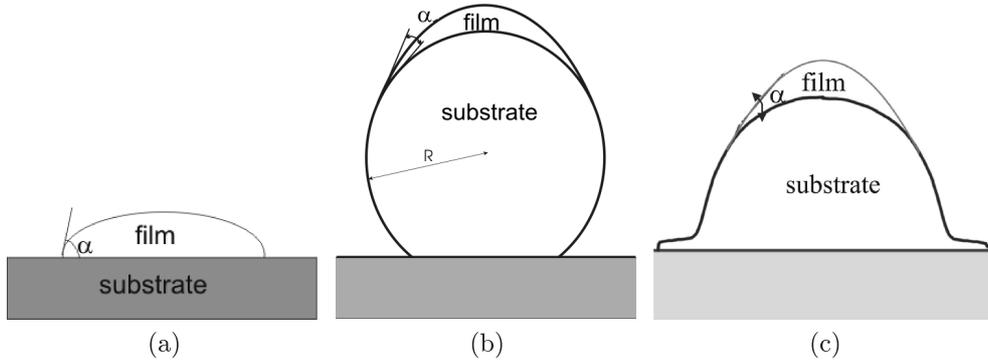


Fig. 1. (a) Planar substrate; (b) Spherical substrate;
(c) Arbitrary axisymmetrical substrate.

The static processes are governed by various parameters, which cause solution bifurcations. The bifurcation points correspond to the stability limit

or loss of stability states. The questions, we are trying to answer in this paper, are: if it is possible to have a stable static equilibrium for all problem parameters: film volume V , wetting angle α , Bond number $Bo = \frac{\rho g R^2}{\sigma}$, where ρ is the film density, g is the gravity acceleration, R is a characteristic length scale, σ is the surface tension between the film and the ambient gas; if the film static equilibrium and its stability depend on the substrate curvature.

2. General problem postulation

We shall consider a general axisymmetric substrate, as shown in Fig. 2. and shall assign to it a cylindrical coordinate system (r, φ, z) . The substrate surface is defined by the radius vector $\mathbf{r}_0 = (r, \varphi, z(r))$ in cylindrical coordinates. Further, an appropriate orthogonal curvilinear coordinate system (φ, η, ξ) fitted to the substrate surface is chosen, such that the substrate surface is the coordinate surface $\xi = 0$. Thus, the surface coordinates are: (φ, η) and the cylindrical coordinates of \mathbf{r}_0 are expressed by them: $r = r(\varphi, \eta)$, $z = z(\varphi, \eta)$.

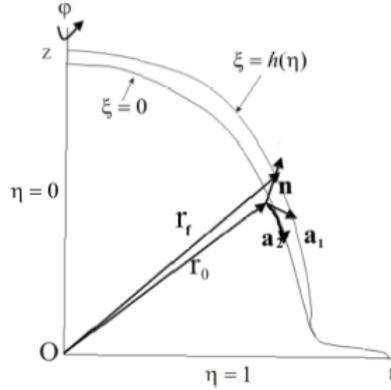


Fig. 2. Sketch of the body fitted coordinate system

The metrics of the substrate surface gives the covariant surface vectors: $\mathbf{a}_1 = (-r \sin \varphi, r \cos \varphi, 0)$, $\mathbf{a}_2 = (r' \cos \varphi, r' \sin \varphi, z')$, where the differentiation is performed with respect to η . The unit normal vector to the substrate is $\mathbf{n} = \left(\frac{-z'}{\sqrt{A}} \cos \varphi, \frac{-z'}{\sqrt{A}} \sin \varphi, \frac{r'}{\sqrt{A}} \right)$, $A = r'^2 + z'^2$ and the substrate mean curvature is $2H = -\frac{r'z'' - z'r''}{A\sqrt{A}} - \frac{z'}{r\sqrt{A}}$. The thin film region, that is adjacent

to the substrate, can be described using $\mathbf{r}_f = \mathbf{r}_0 + \xi \mathbf{n}$, where $r_f = r - \frac{z'\xi}{\sqrt{A}}$, $z_f = z + \frac{r'\xi}{\sqrt{A}}$. The film outer surface $\xi = h(\eta)$ is represented by the radius vector $\mathbf{r}_h = \mathbf{r}_0 + h\mathbf{n}$. Here, we will note that the body fitted coordinates must be applied with a special care far away from the substrate surface [4]. This means that, we assume in our study that the normal vectors in the fluid layer (the film domain) do not intersect.

Since the film is supposed to be at hydrostatic equilibrium, the hydrostatic equation $\nabla \mathbf{p} = -\rho \mathbf{g} \nabla \mathbf{z}$, where p is the pressure, gives: $p = p_0 + p_{at} - \rho g z_f(\eta)$, where p_{at} is the atmospheric pressure and p_0 is an unknown constant.

The Laplace-Young equation (normal stress jump condition) must be fulfilled on the film outer surface, exposed to the gas: $-p_{film} + p_{ambient} = -2H_h \sigma$, where $p_{ambient} = p_{at}$, $p_{film} = p|_{\xi=h(\eta)}$ and $2H_h$ is the mean curvature of the film outer surface:

$$(1) \quad 2H_h = -\frac{r'_h z''_h - z'_h r''_h}{A_h \sqrt{A_h}} - \frac{z'_h}{r_h \sqrt{A_h}}, \quad A_h = (r'^2_h + z'^2_h).$$

The dimensionless forms of the non-linear and the linearized ($h \ll 1$) Laplace-Young equations are respectively:

$$(2) \quad P + B o z + h \frac{B o r'}{\sqrt{A}} + 2H_h = 0,$$

$$(3) \quad P + B o z - \frac{z'}{r \sqrt{A}} - \frac{r' z'' - z' r''}{A \sqrt{A}} - h \left[-\frac{B o r'}{\sqrt{A}} + \frac{z'^2}{r^2 A} + \frac{(r' z'' - z' r'')^2}{A^3} \right] + h' \left[\frac{r' r'' + z' z''}{A^2} - \frac{r'}{r A} \right] - \frac{h''}{A} = 0,$$

where $P = -\frac{R p_0}{\sigma}$ is an unknown parameter.

The boundary conditions for (2) or (3) are the symmetry condition

$$(4) \quad h'(0) = 0,$$

and the contact line condition

$$(5) \quad h(\eta_m) = 0, \quad h'(\eta_m) = -\tan \alpha,$$

where η_m is unknown and α is the prescribed wetting angle. The dimensionless form of the given film volume V is:

$$(6) \quad W = \frac{V}{\pi R^3} = 2 \int_0^{\eta_m} \int_0^{h(\eta)} h_1 h_2 h_3 d\xi d\eta,$$

where $h_1 = r \left| 1 - \frac{2z'\xi}{r\sqrt{A}} \right| + O(\xi^2)$, $h_2 = \sqrt{A} \left| 1 - \frac{2(r'z'' - z'r'')\xi}{A\sqrt{A}} \right| + O(\xi^2)$, $h_3 = 1$ are the Lamé coefficients.

The equation (2) or (3) is a 2^{nd} order Ordinary Differential Equation (ODE) and together with the conditions (4)–(6), it forms a system for the unknowns h , P and η_m . Similarly, P will disappear, if (2) or (3) is differentiated with respect to η , and (2) or (3) becomes a 3^{rd} order ODE, which together with (4)–(6) forms a system for h and η_m .

The non-linear problem (2), (4)–(6) must be investigated, if the wetting angle $\alpha \gg 0$.

3. Planar substrate

The problem of a planar substrate has been treated in [3]. The substrate mean curvature $2H = 0$, if the substrate is flat. The substrate surface is expressed by $\mathbf{r}_0 = (r, \varphi, 0)$ with surface coordinates: (φ, r) and normal vector $\mathbf{n} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ at $A = 1$. The thin film region is $\mathbf{r}_f = \mathbf{r}_0 + z\mathbf{n}$, thus $r_f = r$, $z_f = z$. The film outer surface corresponds to $z = h(r)$ with radius vector $\mathbf{r}_h = \mathbf{r}_0 + h\mathbf{n}$. The curvature of the film outer surface is

$$2H_h = -\frac{h''}{A_h\sqrt{A_h}} - \frac{h'}{r\sqrt{A_h}}, \quad A_h = (1 + h'^2).$$

Then (2)–(6) become:

$$(7) \quad P + Boh - \frac{h''}{A_h\sqrt{A_h}} - \frac{h'}{r\sqrt{A_h}} = 0,$$

$$(8) \quad P + Boh - h'' - \frac{h'}{r} = 0,$$

$$(9) \quad h'(0) = 0,$$

$$(10) \quad h(r_m) = 0, \quad h'(r_m) = -\tan \alpha,$$

where r_m is unknown,

$$(11) \quad W = \frac{V}{\pi R^3} = 2 \int_0^{r_m} h(r) r dr.$$

The solution $h(r)$ of (8)–(11) can be represented as:

$$(12) \quad h_s(s) = h(r) \frac{Bo^{1/2}}{\tan \alpha} = \frac{I_0(s_m) - I_0(s)}{I_1(s_m)},$$

where $s = rBo^{1/2}$, $s_m = r_mBo^{1/2}$ (cf. [3]). The characteristic length R can be chosen such that $R = Bo^{1/2} \left(\frac{V}{2\pi \tan \alpha} \right)^{1/3}$, $s_m^2 = \frac{2I_1(s_m)}{I_2(s_m)}$, which gives $s_m = 2.1075336$.

The film shape of the linearized problem (12) is always convex and independent on α and V , but is applicable only for small $\alpha < 7^\circ$ in order to have $\max h' \leq O(0.1)$.

The non-linear solution $h(r)$ of the non-linear problem (7), (9)–(11) is scaled similarly, but $h_s(s) = h(r)Bo^{1/2}$, i.e., it is dependent on α . Then, the non-linear problem has only a numerical solution. Both linear and non-linear solutions are unique for a given parameter α . The maximum distance dh_s between the linear and non-linear solution is at $s = 0$: $dh_s = 0.00425$ at $\alpha = 15^\circ$ and $dh_s = 0.0208$ at $\alpha = 25^\circ$, that can be seen in Fig. 3.

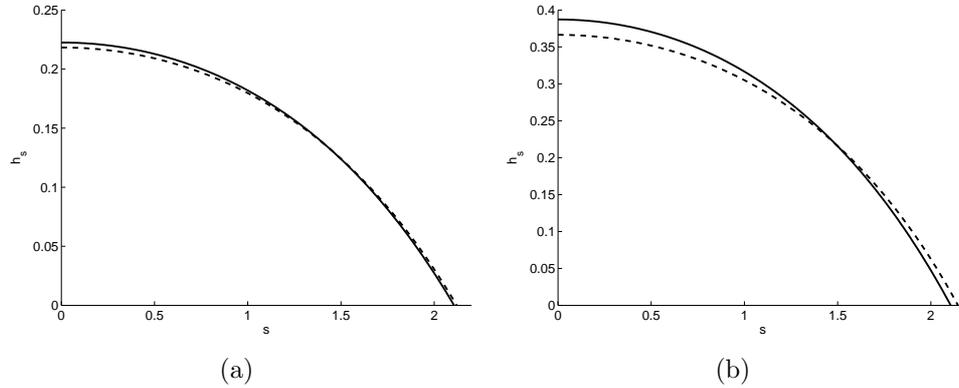


Fig. 3. Comparison between linear (solid line) and non-linear (dashed line) solutions at (a) $\alpha = 15^\circ$; (b) $\alpha = 25^\circ$

4. Spherical substrate

Next, we shall consider a spherical substrate, or a truncated spherical substrate (such forms of substrates appear at ice accretion, cf. [10]). The substrate mean curvature is constant $2H = \frac{2}{R} = \text{const.}$, where R is the sphere radius. Spherical coordinates are used: $(\bar{r}, \varphi, \theta)$, such that the substrate surface is simply given by the radius vector $\mathbf{r}_0 = (R, \varphi, \theta)$ in spherical coordinates. Then the surface coordinates are (φ, θ) and the unit normal vector is: $\mathbf{n} = (\sin \varphi \sin \theta, \cos \varphi \sin \theta, \cos \theta)$ and $A = 1$. The thin film region becomes $\mathbf{r}_f = \mathbf{r}_0 + \xi \mathbf{n}$, where $r_f = (R + \xi) \sin \varphi$, $z_f = (R + \xi) \cos \varphi$. The film outer surface $\xi = h(\theta)$ is given by its radius vector $\mathbf{r}_h = \mathbf{r}_0 + h \mathbf{n}$ and its mean curvature:

$$2H_h = -\frac{(1+h)h'' - 2h'^2 - (1+h)^2}{RA_h^{1.5}} - \frac{h' \cot \theta - (1+h)}{R(1+h)A_h^{0.5}},$$

where $A_h = [h'^2 + (1+h)^2]$.

Then (2), (3) and the boundary conditions become:

$$(13) \quad P + Bo(h \cos \theta + \cos \theta - 1) + 2RH_h = 0,$$

$$(14) \quad P + 2 - 2h - h' \cot \theta - h'' + Bo(h \cos \theta + \cos \theta - 1) = 0.$$

$$(15) \quad h'(0) = 0,$$

$$(16) \quad h(\theta_m) = 0, \quad h'(\theta_m) = -\tan \alpha,$$

$$(17) \quad W = \frac{V}{\pi R^3} = 2 \int_0^{\theta_m} h(\theta) \sin \theta d\theta,$$

where θ_m is an unknown parameter.

The linear problem (14)–(17) has a numerical solution for $Bo \geq 0$, given in [6] and [7], where this problem is solved numerically as an initial value problem by the shooting method (using Runge-Kutta method).

The solution is unique for some values of W at fixed Bo and α , but for others it bifurcates into two different pairs (θ_m, P) , to which two different film shapes (two different solutions h) correspond: one convex shape ($\max(h) = h(0)$), “C” shape, and one “S” shape ($\max(h) \neq h(0)$) or two “S” shapes. The general extremum volume condition (such as for liquid bridge stability, cf. [8],

[9]) is applied in order to obtain the stable film shapes (solutions h) with the stable pairs (θ_m, P) :

$$(18) \quad dW = \frac{\partial W}{\partial P} dP + \frac{\partial W}{\partial \theta_m} d\theta_m = 0,$$

or

$$\frac{\partial W}{\partial P} = 0, \quad \frac{\partial W}{\partial \theta_m} = 0$$

when $dP \neq 0$ and $d\theta_m \neq 0$.

The following maximum volume procedure is constructed in [6]: for fixed α and Bo , the values of P and θ_m are changed monotonously to obtain the set $\Omega = \{W : P_i \leq P \leq P_f, 0 < \theta_m < \theta_1\}$, where P_i and P_f are lower and upper bounds for P , θ_1 depends on the support geometry ($\theta_1 = \pi$ for spherical support, $\theta_1 < \pi$ for truncated spherical support). The maximum volume W_{\max} corresponds to the stability limit point or to the point of stability loss.

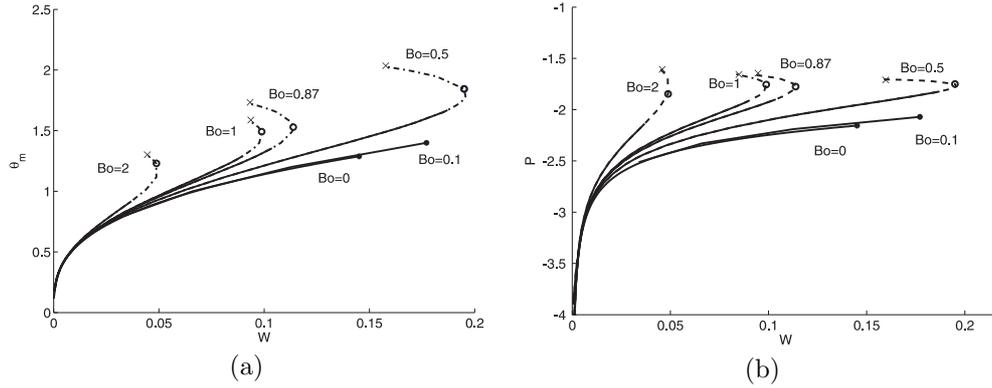


Fig. 4. Plot of the function $W(\theta_m, P)$ at $\alpha = 15^\circ$. The “C” shape branch is with solid line, the “S” shape – with dashed line. The stability limit points are marked by (\circ) and rupture points, $(h(0) < 0)$ by (\times) . The points correspondent to the transformation of thin films into thick $(h > O(0.1))$ are marked by (\bullet)

The stability limit points and the values of the parameters P and θ_m , correspondent to the equilibrium film shapes are shown in Fig. 4 for $\alpha = 15^\circ$ and $0 \leq Bo \leq 2$. We can conclude, according to the analysis performed in [6], that:

- All the “C” shapes are stable, their correspondent points in Ω lie entirely under the stability limit points;
- The “S” shapes are: stable under the stability limit points, unstable above them till reaching the film rupture $(h(0) < 0)$.

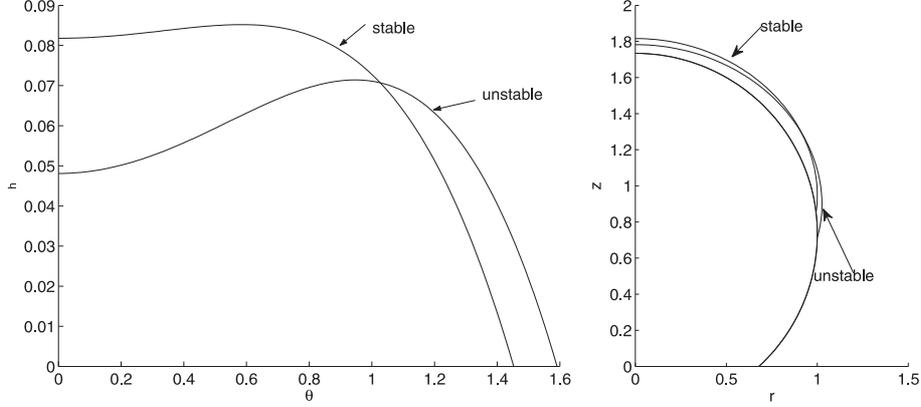


Fig. 5. Different film shapes on a truncated spherical support at $Bo = 0.87$ and $\alpha = 15^\circ$

The stable and the unstable shapes of a thin film on a truncated spherical support are shown in Fig. 5. The parameters are: $0 < \theta_m < 2.395$, $\alpha = 15^\circ$ and $Bo = 0.87$, which corresponds to a film formation on a spherical support with a diameter of 5 mm, for example.

The non-linear problem (13) with (15)–(17) is solved numerically as in the linear case. The linear and the non-linear solutions are almost identical for small α , but are different for large α . For example at $\alpha = 15^\circ$, the comparison between the linear (as in Fig. 4(a)) and the non-linear solutions is presented in Fig. 6. It is well seen that both solutions have bifurcations. The non-linear solution gives smaller stability limit volumes than the ones obtained by the linear solution. The results by the two solutions are almost the same for small $\alpha < 7^\circ$.

5. Arbitrary axisymmetrical substrate

The mean curvature is not constant, $2H \neq \text{const}$, if the support has an arbitrary axisymmetrical shape, described by the equation $z = f(r)$. The substrate surface points have a radius vector $\mathbf{r}_0 = (r, \varphi, f(r))$ in cylindrical coordinates and the correspondent surface coordinates are: (φ, η, ξ) . We take the arc length $s = \int_0^r \sqrt{1 + f'(t)^2} dt$ for the coordinate η . Then $r = r(s)$, $z = f(r(s))$ and $r' = \frac{1}{\sqrt{1 + f_1^2}}$, $z' = \frac{f_1}{\sqrt{1 + f_1^2}}$, where $f_1 = \frac{df}{dr}$. The unit normal vector and

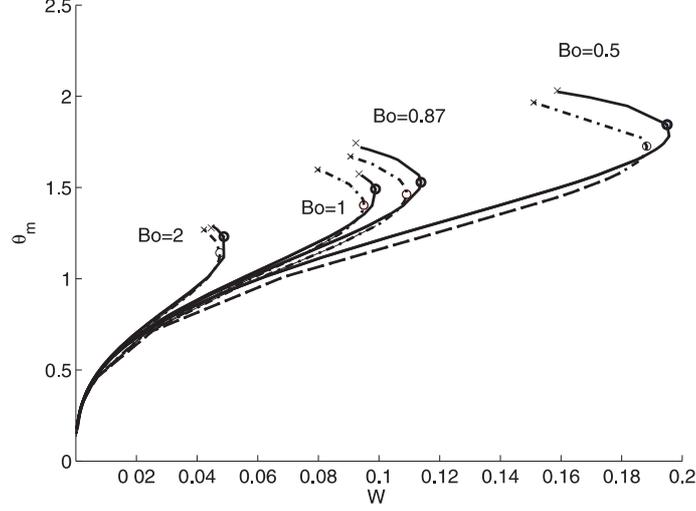


Fig. 6. Comparison between linear (solid line) and non-linear (dash-dotted line) solution. Other notations are as in Fig. 4

the thin film region are respectively: $\mathbf{n} = \left(\frac{-f_1}{\sqrt{B}} \sin \varphi, \frac{-f_1}{\sqrt{B}} \cos \varphi, \frac{1}{\sqrt{B}} \right)$, $A = 1$ and $\mathbf{r}_f = \mathbf{r}_0 + \xi \mathbf{n}$, where $B = (1 + f_1^2)$, $r_f = r - \frac{f_1}{\sqrt{B}} \xi$, $z_f = z + \frac{1}{\sqrt{B}} \xi$. Then, the film outer surface $\xi = h(s)$ is given by $\mathbf{r}_h = \mathbf{r}_0 + h \mathbf{n}$ and $2H_h$ and A_h can be calculated using (1). The linearized equation (3), in the surface coordinates (φ, s) , becomes:

$$(19) \quad P + Bo \cdot f - \frac{f_1}{r\sqrt{B}} - \frac{f_2}{B\sqrt{B}} - h \cdot \left[-\frac{Bo}{\sqrt{B}} + \frac{f_1^2}{r^2 B} + \frac{f_2^2}{B^3} \right] - \frac{h'}{r\sqrt{B}} - h'' = 0,$$

where $f_2 = \frac{d^2 f}{dr^2}$.

The boundary conditions are defined by the equation (4) and by the following relations:

$$(20) \quad h(s_m) = 0, \quad h'(s_m) = -\tan \alpha,$$

$$(21) \quad W = \frac{V}{\pi R^3} = 2 \int_0^{s_m} r h ds,$$

where s_m is unknown.

Often the support has an experimentally registered shape, for example the shape S_3 of Fig. 2 in [10]. Then, the function $f(r)$ can be obtained after a numerical approximation of the experimental points, as it is shown in [7].

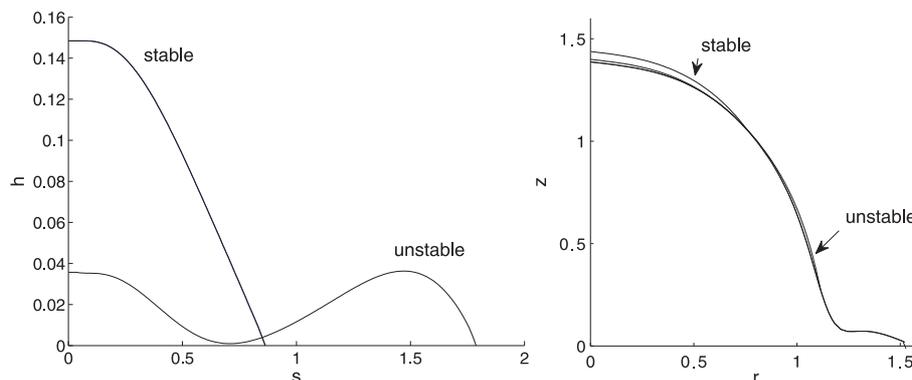


Fig. 7. Different film shapes on an arbitrary support [7] at $Bo = 0.87$, $\alpha = 15^\circ$ and $W = 0.05$

In the case of the considered in [7] substrate shape, a solution bifurcation is again observed for some values of the parameters. An example is shown in Fig. 7. at $Bo = 0.87$, $\alpha = 15^\circ$ and $W = 0.05$. Two different film shapes are seen: convex shape “C” and “S” shape, that correspond to stable and unstable film, respectively.

The solution bifurcation is observed in all performed examples with convex or partially convex (the film spreads only on the convex part of the substrate) substrate shapes. The film rupture starts from the rear part of the film for small α , while for larger α it is moved to the front.

Here, again, the non-linear problem must be studied, if the wetting angle is high enough. However, the results of both the linear and the non-linear problem are different, but qualitatively similar.

6. Conclusion

The present paper is concerned with the static equilibrium problem of a thin film spread on an axisymmetrical substrate. This configuration can be observed, for example, during the drop impact on a substrate formed after the freezing of a previously impacted drop, which can have a splat (planar) form, spherical or truncated spherical form and experimentally registered axisymmetrical form. The stability analysis of the equilibrium film shape under the constraint of volume conservation is performed, at given Bond number

and wetting angle. As a result, it is obtained that the substrate curvature is of major importance for the stability problem: zero mean curvature (planar substrate) creates only convex film shapes and unique solutions; constant (spherical or truncated spherical substrate) or variable mean curvature (arbitrary axisymmetrical substrate) creates convex and “S” shapes and solution bifurcation. It occurs that all convex shapes are stable, some of the “S” shapes are stable, while the others are unstable. The instability branch of the solutions leads to the limiting case of film rupture. In the case of large wetting angles the non-linear equation of film capillary statics must be solved instead of the linear equation.

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